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*PRODUCTION OF NEUTRAL π-MESONS IN VARIOUS NUCLEI BY 260-660 MEV PROTONS**

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The γ-ray yields from the decay of π^0-mesons produced by 260-660 Mev protons in complex nuclei were measured at various angles. The total cross sections for production of π^0-mesons in various nuclei were determined in this energy range.

THE study of processes of production of π^0-mesons in complex nuclei is a natural source of information about their interaction with nuclear matter. This is connected with the fact that, owing to the short life of π^0-mesons in nuclear matter, it is possible to study their interaction with only the nucleons of that nucleus in which they are produced. The results of similar experiments in photoproduction of mesons are much easier to interpret. In the case of bombardment of the nucleus by nucleons the general picture of the phenomena becomes much more complicated, owing to the strong interaction of the incident nucleons with nuclear matter.

In the present work, carried out on the six-meter phasotron of the Joint Institute for Nuclear Research, the yields of γ-rays from the decay of π^0-mesons produced in complex nuclei by protons were measured at various angles. The energy of the protons was varied from 260 to 660 Mev.

METHOD OF MEASUREMENT

The beam of γ-rays from the decay of π^0-mesons produced in the internal target of the accelerator was measured using the telescope described in Ref. 1. The number of protons going through the target was determined by the ionization heating of the target. The arrangement of the experiment and the method of measurement of the absolute cross sections used in the present work were essentially the same as previously described.[1] Therefore we restrict ourselves here to considering only the additional methodological problem connected with measurement of the ratios of differential cross sections for different nuclei.

The differential cross section for γ-ray production is given by the formula

$$d\sigma_A^\gamma / d\Omega = n\varepsilon\,(1 + \varepsilon_{\text{star}}/\varepsilon)\,(1+\delta)\,A / mF_\gamma \Omega N. \tag{1}$$

Here n is the number of γ-rays counted by the telescope per unit time, m is the heat current through the target measured by a thermocouple, F_γ is the efficiency of the γ-telescope, ϵ is the specific ionization loss (for 1 g/cm^2), ϵ_{star} is the heat produced in the target because of star-production, δ is a small correction taking account of the absorption of γ-rays in the target and in air, Ω is the solid angle of counting, A the atomic number of the target material, and N is Avogadro's number. In this work we measured the ratios of the cross sections for various elements to those for carbon (relative cross sections).

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*The results of the present work were reported at the conferences in Moscow (May, 1956) and Geneva (June, 1956).

As the measurements showed, for fixed proton energy and angle of γ-ray counting, the efficiency of the γ-telescope did not depend on the material of the irradiated target. Therefore the relative cross sections were determined by the simple relation:

$$(d\sigma_A^\gamma / d\Omega)' = (n/m)' (1 + \varepsilon_{\text{star}}/\varepsilon)' (1 + \delta)' (A\varepsilon)'. \tag{2}$$

Here, for conciseness, the prime indicates that the quantities occurring in the formula relating to atomic weight A are divided by the corresponding quantities obtained for carbon. The quantities $(1 + \epsilon_{\text{star}}/\epsilon)'$ and $(1 + \delta)'$ constitute small corrections. They are given in Table I. The thickness of the targets used varied from element to element, constituting, on the average, about 1 g/cm^3.

TABLE I

Element	$(1+\delta)'$	$(1+\varepsilon/\varepsilon_{\text{star}})'$ calculated	$(1+\varepsilon/\varepsilon_{\text{star}})'$ exper.
Li_6	1.00	1.01	
Li_7	1.00	1.01	
Be	1.00	1.00	
C	1.00	1.00	1.00
Al	1.01	1.03	1.04±0.03
Cu	1.02	1.03	
Cd, Sn	1.04	1.04	
Pb	1.06	1.05	1.06±0.03
U	1.08	1.05	

The second column of the table gives corrections calculated to an accuracy of 1 — 2% on the basis of the spectra of the particle-ray stars measured by Grigor'ev and Solov'eva.[2] An experimental determination of the ratio $(1 + \epsilon_{\text{star}}/\epsilon)'$ was also attempted. For this, targets prepared from graphite, aluminum and lead were placed in a magnetic channel of the phasotron and irradiated by the proton beam. A very sensitive small-gauge thermopile (of sensitivity 3 × 10^{11} Mev/sec.mv) containing 300 copper-constantan thermocouples was used to measure the small heat current through the target. The relative proton currents through the targets were determined in these experiments by the activation of thin copper foils. It should be noted that a similar experiment cannot be carried out on the internal circulating beam of the accelerator because in this case the method of activation of the foil does not furnish the necessary high accuracy of measurement of the relative proton beams through the target. Because the intensity of the extracted beam is small in comparison with the internal beam, it was necessary to use thick targets (≈ 1 cm). Because of this the time of establishing thermal equilibrium was large (≈ 1 hr) which restricted the possibilities of accurate measurement. The results of measurement are given in Table I (final column).

INVESTIGATION OF THE ANGULAR DISTRIBUTION OF γ-RAYS

The measurements made earlier[3] showed that the relative cross sections for production of γ-rays at zero angle grew with increasing atomic weight sustantially more slowly than at 180°. This means that the angular distribution of π^0-mesons depends substantially on the atomic weight of the target material and, consequently, in order to obtain the values of the total cross sections for production of π^0-mesons in various elements, it is necessary to have information about the angular distributions of the γ-rays.

In the present work the γ-ray yields were measured in the angular intervals 0 — 33° and 147 — 180° in the laboratory system (l.s.). In Table II the results of these measurements, carried out for a proton energy of 660 Mev, are given in the effective center-of-mass system (c.m.s.) of the colliding nucleons.

The velocity $\overline{\beta}_c$ = 0.50 of the effective c.m.s. was found by averaging the velocities of the c.m.s. of the colliding nucleons under the assumption that the momentum distribution of the nucleons in the nucleus is Gaussian with standard deviation 120 Mev/c (corresponding to an energy of 8 Mev) cut off at 22 Mev. The quantity $\overline{\beta}_c$ is insensitive to the form of the momentum distribution for a proton energy of 660 Mev. Thus, if a Fermi distribution with Fermi energy 22 Mev is taken for the nucleons in the nucleus, then $\overline{\beta}_c$ = 0.48. These values of $\overline{\beta}_c$ are not very different from the velocity of the c.m.s. for the case of collision with a nucleon at rest (β_c = 0.52).

Since the velocity of the π^0-mesons is near to that of light, the angular distributions of γ-rays practically coincide with the angular distributions of the π^0-mesons from whose decay they arise.

The decrease in the relative small-angle yields of π^0-mesons with increasing atomic weight can be explained by the fact that in this case mainly γ-rays are registered from the decay of π^0-mesons produced in that part of the atom which is screened from the bombarding protons by the nucleons. In going through the nucleus, the protons lose energy in collisions with nucleons of the nucleus and do not take part in the process of meson production. With increasing atomic weight the screening, naturally, increases. As the angle of observation increases, the effect of screening weakens, as can be seen from Table II.

In the lightest elements the lack of transparency for protons and the absorption of mesons appear to be comparatively weak. From Table II it can be seen that the angular distribution of γ-rays for these nuclei

TABLE II

ϑ° c.m.s.	0	35	55	160	169	180
Li_6	0.80±0.04	—	0,91±0,03	1,00±0,07	—	1 ,00±0,04
Li_7	0,74±0,03	—	0.82±0,03	1.00±0,06	0,96±0,10	1,00±0,04
Be	0,70±0,02	—	0.80±0,04	1,10±0,07	—	1,00±0,03
C	0.61±0,02	0,63±0,04	0,68±0,03	1,02±0,05	0,95±0,07	1,00±0,02
Al	0,50±0.02	—	0.58±0,04	0,97±0.06	—	1,00±0,03
Cu	0,37±0,02	—	0,48±0,03	0,92±0,07	—	1,00±0,03
Cd, Sn	0,31±0,01	—	0,42±0,04	—	—	1,00±0,03
Pb	0,29±0,01	—	0,38±0,02	0,95±0,04	—	1,00±0,03

corrected for the small effect of screening are practically isotropic. From this it is possible to draw the conclusion that the angular distribution of π^0-mesons produced at 660 Mev in proton-nucleon collisions are nearly isotropic. This agrees with other data.[1,4]

In order to find out how the angular distribution of π^0-mesons changed with decreasing the energy in the nucleon collisions, relative yields of γ-rays for angles of 33 and 147° in the l.s. for lithium were compared for various proton energies. The changes demonstrated that the anisotropy of the π^0-meson angular distribution grew with decreasing energy. If the π^0-mesons produced in p-n collisions at 450 Mev are taken to be distributed as $1/3 + b\cos^2\vartheta$, then it follows from our measurements that $b \geq 0.5 \pm 0.2$. This result agrees with the data obtained by studying γ-ray spectra.[4] Deviations of the π^0-meson angular distribution from isotropy become noticeable for energies below 550 Mev.

The change in relative yields of γ-rays for the angles noted above also continues for energies below 400 Mev. However, a quantitative interpretation of these measurements is not possible in practice because here the yields depend very essentially on assumptions about the character of the momentum distribution of nucleons in the nucleus, and about the spectrum of π^0-mesons.

DEPENDENCE OF THE DIFFERENTIAL CROSS SECTIONS FOR PROTONS OF VARIOUS ENERGIES ON ATOMIC WEIGHT

The dependence of the differential cross sections for γ-ray production on the atomic weight of the target material was measured for proton energies of 260, 340, 445 and 660 Mev at two angles. In varying the energy the angles changed somewhat, being 33 and 147° (l.s.) at 660 Mev and 40 and 140° at 260 Mev. The relative cross sections obtained are given in Table III (angles 33 — 40°) and Table IV (angles 147 — 140°).

TABLE III

E_p(Mev)	660	445	340	260
D*	0.48±0.05	0.30±0.06	—	—
Li_6	0,70±0.15	0,75±0.15	—	—
Li_7	0,89±0.02	0,88±0.02	0.72±0.03	0.72±0.03
Be	0,95±0.03	0,88±0.03	0.85±0.05	0,77±0.04
C	1,00	1,00	1.00	1.00
Al	1.48±0.04	1.52±0.04	1.67±0.07	1,82±0.11
Cu	2.3 ±0,1	2.8 ±0.2	2,9 ±0.1	3,5 ±0.2
Sn	3.2 ±0.1	3.8 ±0.1	4.0 ±0.2	4.9 ±0.3
Pb	4,2 ±0.1	5,0 ±0.2	5.7 ±0.3	6.3 ±0.2
U	4,1 ±0,3	5.2±0.3	—	—

*Taken from Ref. 1.

The angular distribution of the π^0-mesons ($F_\pi(\vartheta)$) and that of the γ-rays produced in their decay ($F_\gamma(\vartheta)$) are connected by the integral relation

$$F_\gamma(\vartheta) = \int_{-1}^{+1} F_\pi(\vartheta_0)\, G(\beta, \cos\vartheta, \cos\vartheta_0)\, d\cos\vartheta_0. \qquad (3)$$

Here β is the velocity of the π^0-meson,

$$G(\beta, \cos\vartheta, \cos\vartheta_0) = {}^1/_2\,\gamma^{-2}(1 - \beta\cos\vartheta\cos\vartheta_0) \times$$
$$\times\, [(\beta\cos\vartheta_0 - \cos\vartheta)^2 + \sin^2\vartheta/\gamma^2]^{-3/2}, \quad \gamma^2 = 1/(1-\beta^2).$$

Equation (3) shows that the angular distribution $F_\gamma(\vartheta)$ differs slightly from $F_\pi(\vartheta)$ only for $\beta \gtrsim 0.9$. The degree of anisotropy in the distribution $F_\gamma(\vartheta)$ decreases rapidly with decreasing β. Thus, even if the angular distribution of π^0-mesons does not change with decreasing proton energy, owing to the decrease in velocity of the π^0-mesons, the difference between the angular distributions of the γ-rays, which occur for light and heavy nuclei for energies of 450 — 660 Mev, should vanish as the threshold for meson production is approached. Therefore, with decreasing proton energy the relative cross sections measured at 33° for heavy nuclei should increase, and at 147°, decrease. As can be seen from Tables III and IV, this drawing together of the values of the relative cross section really does occur; however, simultaneously with this, the relative cross sections at 147° not only does not decrease, but even systematically grows as the threshold for meson production is approached. This indicates that the angular distribution of π^0-mesons changes in the energy interval 260 — 660 Mev. The latter may be connected with a diminishing of the already discussed effect of screening, brought about by the decrease in the meson production cross section. A quantitative evaluation of this change in angular distribution of π^0-mesons, which requires extensive calculation (for example, by

TABLE IV

E_p (Mev)	660	445	340	340**	260
D*	0.30±0.05	0.20±0.05	—	0.143±0.004	
Li_6	0.50±0.10	0.55±0.10	—	—	—
Li_7	0.75±0.01	0.75±0.02	0.64±0.02	0.57±0.02	0.55±0.02
Be	0.83±0.03	0.79±0.03	0.82±0.04	0.83±0.02	0.74±0.03
C	1.00	1.00	1.00	1.00	1.00
Al	1.70±0.04	1.84±0.03	1.96±0.06	2.1±0.1	1.9±0.1
Cu	2.9 ±0.1	3.6 ±0.1	4.2 ±0.2	4.2±0.2	—
Cd	4.6 ±0.1	5.7 ±0.2	6.4 ±0.3	—	—
Pb	7.1 ±0.2	8.2 ±0.4	9.6 ±0.5	7.8±0.4	9.4±0.5
U	6.6 ±0.6	—	—	—	—

* Taken from Ref. 1.

** Taken, for comparison, from Ref. 5, where the measurements were carried out at 135° in the l.s.

the Monte Carlo method), has not been carried out in the present work.

Comparison of results for measurements of the dependence of the yield of charged mesons on atomic weight with the data on the absorption of π^0-mesons in nuclei allowed several authors[6,7] to conclude that there is a substantial decrease in the probability of π-meson production for the strongly bound nucleons in the interior of the nucleus. Elucidation of the analogous problem in the production of π-mesons by nucleons is also of considerable interest. However, the strong interaction of the incident nucleons with nuclear matter already discussed, leads by itself to the situation in medium and heavy nuclei where the π-mesons are effectively produced only in the surface of the target nucleus. If a decrease in the probability of π-meson production inside the nucleus really does take place, then this should be reflected in the behavior of the relative cross sections only in the region of light nuclei, and only in the case when the absorption of the π-mesons leaving the nucleus is small. In connection with this, a definitive answer to the question about the existence of a supplementary mechanism which diminishes the probability of meson production by protons on the strongly bound nucleons inside the nucleus is possible only after carrying out calculations which take the absorption of protons and mesons in light nuclei correctly into account. However, the existence of models of the complex nucleus which could be used to solve this problem, have a sufficiently reliable basis only for the case of heavy nuclei. Therefore, from the presently-available experimental data on the production fo π^0-mesons in light nuclei (Refs. 3, 5, 8 and this work) the relative probabilities of π^0-meson production in the surface and on the internal nucleons of the nucleus cannot be determined.

DEPENDENCE OF THE TOTAL CROSS SECTIONS ON THE PROTON ENERGY

Using the data about the γ-ray angular distribution given in Table I, and the value of the absolute differential cross section determined for energy 660 Mev in the work of Ref. 1, we find that the total cross section for production of π^0-mesons by 660 Mev protons on carbon is

$$\sigma_C^{\pi^0} = (28 \pm 3) \cdot 10^{-27}\ \text{cm}^2.$$

The total cross sections for other elements are given in Fig. 1.

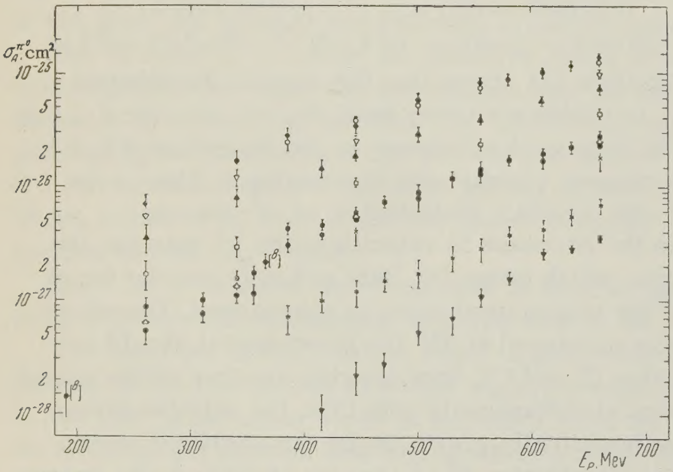

FIG. 1. Total cross sections for production of π^0-mesons by protons on various nuclei ○ — U, ◆ — Pb, ▽ — Cd, Sn, ▲ — Cu, □ — Al, ■ — C, △ — Be, ● — Li_7, × — Li_6, ✱ — D (from Ref. 1), ▼ — H (from Ref. 1).

The dependence of the total cross section on energy was measured for a series of elements (see Fig. 1). In these experiments the variation in proton energy was achieved by inserting targets at different distances from the center of the accelerator; γ-rays were counted at angles 33 and 147°. The change in angular distribution of the π^0-mesons was taken into account in the determination of the value of the total cross sections.

As was pointed out above, the change in angular distribution of the π^0-mesons connected with the effects of screening and absorption was small in the case of the lightest nuclei. This permits the analysis of the energy dependence of the total cross section obtained for a light element (lithium) to be carried out without taking into account the change in transparency of the nucleus. Comparison of the energy dependence of the total cross section measured near the threshold for meson production with calculated curves permits us to draw (in the framework of the impulse approximation) several conclusions about the character of the energy distribution of nucleons in the nucleus since the behavior of the calculated curves in this energy region is, in large part, determined by the form of the energy distribution of the nucleons. The excitation function is given (in relative units) in Fig. 2, together with curves calculated in the impulse approximation for the case of a Gaussian distribution of nucleons in the nucleus with standard deviation 120 Mev/c (nucleon energy 8 Mev), cut off at an energy of 22 Mev. The Pauli Principle was not taken into account in the calculation since the nucleon energies after meson production substantially exceeded the Fermi energy. The divergence of the energy dependencies of the cross section from the calculated curves serves to indicate that there should be more high energies in the energy spectrum of the nucleons inside the nucleus than occur in the Gaussian distribution taken.

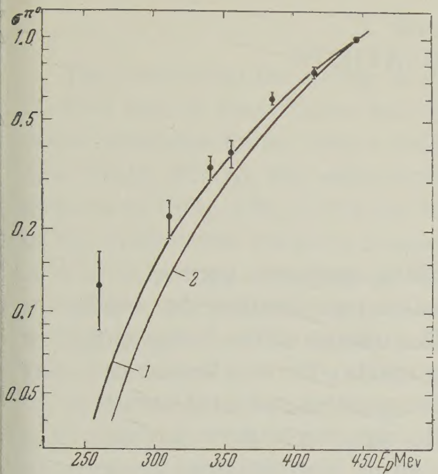

FIG. 2. Energy dependence of the cross section for production of π^0-mesons in lithium near threshold. Curve 1 corresponds to the dependence of the cross section production of π^0-mesons in collisions of the incident protons with nucleons in the nucleus: $\sigma^{\pi^0} \sim p_{\max}^3$; curve 2 was calculated for the case $\sigma^{\pi^0} \sim p_{\max}^4$, where $p_{\max}$ is the maximum momentum of the π^0-meson in the c.m.s. of the colliding nucleons.

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RELAXATION POLARIZATION AND LOSSES IN NONFERROELECTRIC DIELECTRICS WITH HIGH DIELECTRICS CONSTANTS

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It is shown that in polycrystalline dielectrics of the $SrTiO_3 - Bi_2O_3 \cdot nTiO_2$ systems, over a wide range of concentration of the components, there occurs a relaxation polarization that leads to a very high dielectric constant without ferroelectric properties. The nature of the relaxation polarization changes upon change of the relative amounts of the components. Three classes of strontium-bismuth titanates have been established, which differ in composition, crystal structure, and properties. Dielectrics of the first class have the perovskite structure; they may be divided into two groups that have different types of relaxation polarization. For the first group, as was shown earlier,[1] the dielectric constant is close to 1000; for the second group ϵ reaches 6000 over a quite broad frequency range. The frequency and temperature dependence of ϵ and of tan δ are studied in relation to the composition and structure of the new dielectrics. The experimental results are interpreted on the basis of ideas about ionic relaxation polarization.

1. STATEMENT OF THE PROBLEM

It was shown by Skanavi and Matveeva[1] that the conjunction of two factors — an internal field favoring polarization, which results from the perovskite structure, and a large ionic relaxational polarizability, which occurs in the presence of vacant sites in the lattice — leads to a very high dielectric constant without ferroelectric properties.

The object of the present research was a more detailed experimental study of the laws of relaxational polarization, not only for the strontium-bismuth titanate (SBT-1) studied in Ref. 1, but also — as far as possible — for all dielectrics of the $SrTiO_3 - Bi_2O_3 \cdot nTiO_2$ systems, with variation of the relative amounts of the components. Our intention was to trace the influence of changes of crystal structure in general, and of the number of vacant sites in particular, on the nature of the relaxational polarization and on the magnitude of the fundamental characteristics (dielectric constant, loss angle, etc.). Such an experimental study was of interest both for confirmation of some of the ideas of Ref. 1, and for elucidation of methods of obtaining new dielectrics with specified properties.

2. SAMPLES AND METHODS OF INVESTIGATION

The samples studied for this purpose were polycrystalline dielectrics of the systems $SrTiO_3 - Bi_2O_3 \cdot 2TiO_2$, $SrTiO_3 - Bi_2O_3 \cdot 3TiO_2$, and $SrTiO_3 - Bi_2O_3 \cdot 4TiO_2$, with wide variation of the relative amounts of the components. They were progressively synthesized at temperatures 1000, 1100, 1200, and 1300° C and were finally fired until there were no exposed pores. The structure of these dielectrics was studied with x-rays, their densities and the numbers of atoms in the unit cell were determined, and the dielectric constants ϵ and loss tangents tan δ were measured as functions of temperature and frequency. For the density measurement, the samples were pulverized in an agate mortar until the particle dimensions were 2 to 7 μ; this was known to be less than the mean dimensions of the air spaces in the samples. The density was determined by means of a pycnometer 5 cm^3 in volume; toluene was used, after preliminary evacuation. The crystal lattice parameters were determined from a Debye pattern taken with a camera 114.6 mm in diameter. An asymmetric method was used, with copper radiation and a nickel filter. This was done by one of the authors and also, at our request, by N. A. Porai-Koshits; the results of the two determinations agreed. The relative intensities were measured with roentgenograms automatically traced on a URS-50I ionization assembly.

The values of ϵ and tan δ were measured over a wide range of temperatures and frequencies by means of standard bridge and resonance apparatus. The measurements in the frequency interval 40 to 180 Mc were of preliminary nature, being subject to comparatively large errors.

3. EXPERIMENTAL RESULTS

A. Structural Properties of Strontium-Bismuth Titanates

The results of the x-ray analysis show that all the strontium-bismuth titanate systems studied may be divided into at least three basic classes. In the first class, with high concentrations of strontium titanate, solid solutions form, with a cubic lattice of perovskite type. In the system $SrTiO_3 - BiTiO_{7/2}$, or $SrTiO_3 - 1/2\ (Bi_2O_3 \cdot 2TiO_2)$, the solid-solution range is bounded by a concentration of 25 to 28 mol percent, in the system $SrTiO_3 - Bi_{2/3}TiO_3$ by 35 mol percent, and in the system $SrTiO_3 - Bi_{1/2}TiO_{11/4}$, by 30 mol percent, of the respective bismuth titanates. In the second class (with high concentrations of bismuth titanate) there is a broad range of mixtures of a solid solution of perovskite type with a new crystalline phase of complex structure, corresponding to chemical compounds of strontium titanate with bismuth titanate (for example, $SrBi_2Ti_4O_{12}$). For the third class, with small concentrations of $SrTiO_3$ (less than about 30 mol percent), the Debye patterns are very complicated; they contain, together with other lines, the lines of bismuth titanate.

In the solid-solution range (class 1), for all three systems, the lattice has a cubic structure with the parameter $a = 3.898 \pm 0.002$ A. In these systems there is a possibility of formation of two types of solid solution: a substitutional solid solution, and a solid solution with a structural defect of subtractive type. The radii of the bismuth and strontium ions are very nearly equal (~ 1.2 A); it is therefore to be expected that a bismuth ion in a solid solution will be able to occupy sites only in the strontium portion of the lattice, in which case the lattice constant need not change. For formation of a substitutional solid solution, a reaction must proceed, in which oxides dissociate with liberation of gaseous oxygen and lowering of the valence of the Bi or Ti; for instance, when n = 2 the scheme is the following:

$$x\,SrTiO_3 + (1-x)\,BiTiO_{7/2} \rightleftarrows (Sr_xBi_{1-x})TiO_3 + {}^1/_4\,(1-x)\,O_2\uparrow + (1-x)\,(e), \tag{1}$$

where x is the molecular concentration of $SrTiO_3$, and where (e) is a quasi-free electron that results from liberation of oxygen. It is clear that such a reaction is promoted by the high firing temperature (above 1450°C) that is necessary for complete sintering of materials containing less than 10 molar percent of bismuth titanate. At lower sintering temperatures (≤ 1450°C, materials with high concentration of bismuth titanate), dissociation of oxides is less probable, and subtraction solid solutions may form.

Depending on the number n in the formula $Bi_2O_3 \cdot nTiO_2$, vacant sites may form in the strontium part of the crystal lattice, in the strontium and titanium parts, or in the strontium and oxygen parts, in accordance with the following schemes: for n = 2, for example,

$$x\,SrTiO_3 + (1-x)\,BiTiO_{7/2} \rightarrow (Sr_xBi_{1-x})\,TiO_3 \cdot {}^1/_6\,(1-x)\,Sr^*Ti^*O_3, \tag{2}$$

where the asterisk after the symbol for the element represents a corresponding vacant site; for n = 3,

$$x\,SrTiO_3 + (1-x)\,Bi_{2/3}TiO_3 \rightarrow (Sr_x\overset{*}{Sr}_{1/3\,(1-x)}\,Bi_{2/3\,(1-x)})\,TiO_3; \tag{3}$$

and for n = 4,

$$x\,SrTiO_3 + (1-x)\,Bi_{1/2}TiO_{11/4} \rightarrow (Sr_x\overset{*}{Sr}_{1/2\,(1-x)}\,Bi_{1/2\,(1-x)})\,Ti\,(O_{1/4\,(11+x)}O^*{}_{1/4\,(1-x)}). \tag{4}$$

To test whether vacant sites are actually formed in the crystal lattice when the sintering temperature is not very high, the number of atoms in the unit cell was determined from the experimental data on density and lattice parameter, by the known formula

$$N = Vd/A \cdot 1.65 \cdot 10^{-24}, \tag{5}$$

where d is density, V is the volume of the unit cell, and A is the mean atomic weight, calculated from the proportions of the materials [left sides of formulas (2) to (4)]. In Table I the number of atoms in the unit cell, obtained in this way, is compared with the number of atoms in the unit cell, directly calculated from the right sides of formulas (2) to (4).

The data of Table I show that the number of atoms in the unit cell of the solid solutions decreases with increase of the concentration of bismuth titanate. This indicates the formation of a solid solution of the second kind, of subtractive type; however, the decrease proceeds more rapidly than it should according to the right side of formulas (2) to (4). This makes it impossible to determine the type of vacant sites.

Therefore for determination of the most probable type of vacant sites, a roentgenographic method was used. Because of the large difference between the atomic scattering factors of Sr^{2+} and Bi^{3+}, one can infer the type of solid solution from the change in the intensity of the reflections. Calculations of the relative intensities were carried out for the space group $O'_h - P_m3m$ for all three systems, under the condition that the Bi ions occupy sites in the strontium part of the lattice and that the number of vacant sites for each system is determined by the right side of formulas (2) to (4). Table II gives a comparison of the relative intensities of various reflections, as obtained experimentally, with the calculated values, for the compositions with the largest concentrations of bismuth titanate among those listed in Table I.

TABLE I

System	Composition	Density g/cm³	Number of atoms in unit cell	
			From (5)	Calc. by (2)-(4)
$SrTiO_3-Bi_2O_3\cdot 2TiO_2$	$SrTiO_3$	5.04	4,97	5.0
	$0.9\ SrTiO_3+0.1\ BiTiO_{7/2}$	5.31	4,90	4,975
	$0,8\ SrTiO_3+0.2\ BiTiO_{7/2}$	5.46	4,80	4.943
	$0,7\ SrTiO_3+0,3\ BiTiO_{7/2}$	5.54	4.67	4,920
$SrTiO_3-$ $Bi_2O_3\cdot 3TiO_2$	$0.9\ SrTiO_3+0.1\ Bi_{2/3}TiO_3$	5,19	4,90	4,966
	$0,8\ SrTiO_3+0.2\ Bi_{2/3}TiO_3$	5.32	4,83	4,933
	$0,7\ SrTiO_3+0,3\ Bi_{2/3}TiO_3$	5.36	4,73	4.900
$SrTiO_3-$ $Bi_2O_3\cdot 4TiO_2$	$0.7\ SrTiO_3+0,3\ Bi_{1/3}TiO_{11/4}$	5,25	4.79	4.775

TABLE II

hkl	$I_{hkl} : I_{200}$					
	$0.7\ SrTiO_3+0,3\ BiTiO_{7/2}$		$0.7SrTiO_3+0.3Bi_{2/3}TiO_3$		$0.7SrTiO_3+0,3Bi_{1/2}TiO_{11/4}$	
	Calc.	Exptl.	Calc.	Exptl.	Calc.	Exptl.
100	31	17	19	17	9	19
110	227	236	205	196	191	228
111	78	72	70	61	64	68
200	100	100	100	100	100	100
210	17	13	9,7	7.7	5,6	8,1
211	68	82	77	76	71	92
220	59	40	59	46	59	58
300,221	8,5	4.0	5.0	5.6	2,5	6,5
310	36	32	32	27	30	34
311	24	8,5	19	9.0	19	8.6
222	19	16	19	22	18	16
321	48	37	43	42	40	39
400	11	6.3	11	9	11	11
300,411	34	22	30	22	29	30

The intensity of the (200) reflection was taken as 100% intensity; it had the largest structure factor. Among the various reflections, the ones that are most sensitive to changes of intensity, upon replacement of strontium by bismuth and formation of vacant sites, are those for which the structure amplitudes of the titanium and oxygen have a sign opposite to that of the structure amplitudes of the strontium and bismuth. In Table II, such reflections are (100), (210), and (300, 221). Compariso of the calculated and experimental intensities of these three most "sensitive" reflections shows that the best agreement occurs for the solid solution of composition $0.7SrTiO_3 + 0.3Bi_{2/3}TiO_3$, corresponding to the $SrTiO_3 - Bi_2O_3{\cdot}nTiO_2$ system in which vacant sites should form only in the strontium part of the lattice.

However, as Table II shows, we may infer that a similar situation occurs also in the other systems at concentrations that correspond to the range of solid solutions of subtractive type (group a of structural class 1). In fact, the experimentally determined intensities of the "sensitive" reflections for the systems $SrTiO_3 - Bi_2O_3{\cdot}2TiO_2$ and $SrTiO_3 - Bi_2O_3{\cdot}4TiO_2$ differ little (30% or less) from the calculated intensities of the same reflections for the system $SrTiO_3 - Bi_2O_3{\cdot}3TiO_2$. Therefore, for example, upon sintering of samples of the system $SrTiO_3 - Bi_2O_3{\cdot}2TiO_2$ there must occur a liberation (and escape) of surplus Bi_2O_3. Just this phenomenon is observed experimentally and is confirmed by the results of a chemical analysis, made at our request by E. D. Borneman and O. A. Alekseeva.*

Thus we may expect the nature of the polarization of dielectrics of all three systems, within the bounds of group a of structural class 1, to be similar; for we may suppose with a high degree of probability tha in these systems solid solutions of subtractive types form, with vacant sites in the strontium part of the lattice.

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*The authors express their gratitude to E. D. Borneman and O. A. Alekseeva for this work.

B. Dielectric Properties of Dielectrics of the Systems $SrTiO_3 - Bi_2O_3{\cdot}nTiO_2$

Some of the results of numerous measurements are given in Figs. 1 to 5.

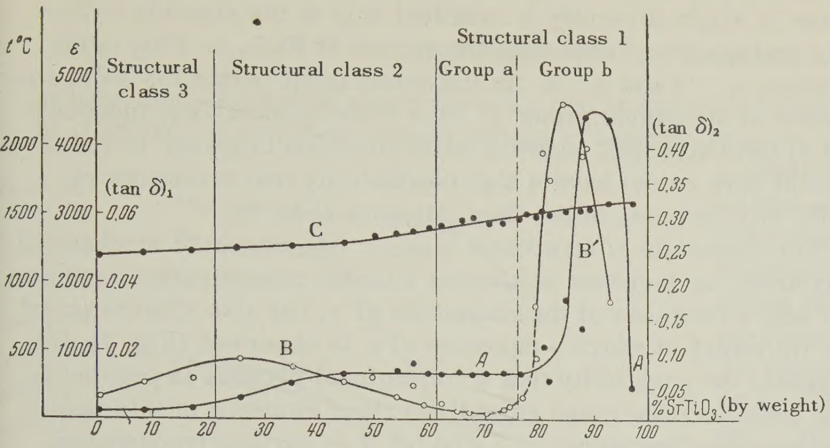

FIG. 1. Dependence of ϵ (curve A) and tan δ (curve B, scale 1; curve B′, scale 2), at room temperature and at frequency 1.5 megacycles/sec, and also of sintering temperature (curve C), on $SrTiO_3$ content, in a batch of dielectrics of the system $SrTiO_3 - Bi_2O_3{\cdot}2TiO_2$.

From Fig. 1 it is evident that for n = 2, the dielectric constant and loss angle at frequency 1.5 megacycles and room temperature attain widely different values, according to the structure of the materials obtained. For the first structural class, the dielectric constant is about 1000 at concentrations of $Bi_2O_3{\cdot}2TiO_3$ greater than 15% by weight (group a); it reaches 5000 to 6000 at smaller concentrations of $Bi_2O_3{\cdot}2TiO_2$ (group b).

The loss angle is inconsequential for materials of group a and increases abruptly for materials of group b. The increase of ϵ and tan δ in materials of group b is accompanied by a change of color from light cream (group a) to dark gray (group b). For structural classes 2 and 3, the the dielectric constant decreases markedly with decrease of $SrTiO_3$ content, and tan δ undergoes an increase. Similar dependence of ϵ and tan δ on concentration is observed in the other systems (n = 3, n = 4). From Fig. 2 it is evident that the polarization of the solid solutions of the system $SrTiO_3 - Bi_2O_3{\cdot}3TiO_2$ (as is also true of the other systems) is of relaxational type. With increase of the concentration of bismuth titanate, the temperatures at the maxima of ϵ and of tan δ increase (curves 1, 2, and 3 of Fig. 2). For the first structural class (large concentrations of bismuth titanate, curves 4 and 5 of Fig. 2), ϵ decreases with increase of the bismuth titanate content, but the temperature of maximum ϵ remains practically unchanged. Fig. 3 illustrates the clearly expressed relaxational nature of the polarization of these solid solutions. It is clear that with increase of frequency, the temperatures of maximum ϵ and tan δ for a given material increase systematically. At sufficiently high temperatures, the values of ϵ for different frequencies agree.

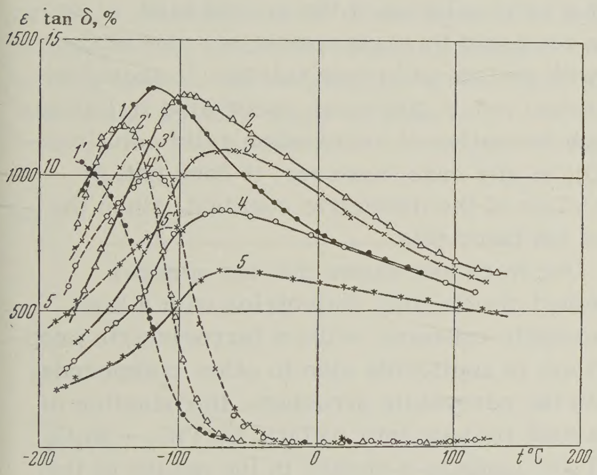

FIG. 2. Dependence of ϵ (curves 1 — 5) and tan δ (curves 1′ — 5′) on temperature for dielectrics of the system $SrTiO_3 - Bi_2O_3{\cdot}3TiO_2$, at frequency 1000 cps. 1 and 1′, dielectric of composition 0.9 $SrTiO_3$ + 0.1 $Bi_{2/3}TiO_3$; 2 and 2′, dielectric of composition 0.8 $SrTiO_3$ + 0.2 $Bi_{2/3}TiO_3$; 3 and 3′, dielectric of composition 0.7 $SrTiO_3$ + 0.3 $Bi_{2/3}TiO_3$; 4 and 4′, dielectric of composition 0.6 $SrTiO_3$ + 0.4 $Bi_{2/3}TiO_3$; 5 and 5′, dielectric of composition 0.5 $SrTiO_3$ + 0.5 $Bi_{2/3}TiO_3$.

The polarization retains its relaxational character also for solid solutions with small bismuth titanate content (for example, group b for n = 2, Fig. 4). However, the temperatures of maximum ϵ and tan δ change faster with increase of frequency than is the case at larger concentrations of bismuth titanate.

The value of ϵ for solid solutions of group a, at room temperature, remains practically unchanged up to frequency 5×10^7 cps; it decreases at higher frequencies. The loss angle of dielectrics of this group rises sharply in the frequency range 10^6 to 10^8 cps. Upon increase of the concentration of bismuth titanate in the solid solutions, the frequency corresponding to the sharp rise of tan δ decreases. At bismuth titanate concentrations below 10 mol percent (group b), the dispersion of ϵ and the maximum of tan δ are located in a lower frequency range (Fig. 5).

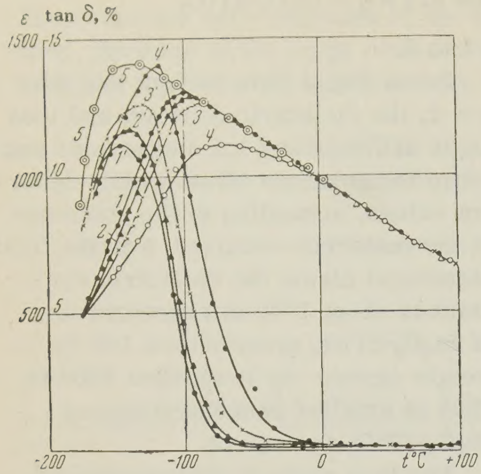

FIG. 3. Dependence of ε (curves 1 — 5) and tan δ (curves 1′ — 4′) of solid solution of composition 0.7 $SrTiO_3$ + 0.3 $Bi_{2/3}TiO_3$ on temperature for various frequencies. 1 and 1′, frequency 4 × 10^2 cps; 2 and 2′, frequency 10^3 cps; 3 and 3′, frequency 5 × 10^3 cps; 4 and 4′, frequency 2.5 × 10^5 cps; 5, frequency ~ 0.2 cps. At frequency ~ 0.2 cps and for temperatures above 0°C, there begins an increase of ε with temperature, not shown in the figure, due apparently to high-voltage polarization.

The experiments showed, as was expected, that the polarization of dielectrics of group a of structural class 1 for all three systems (n = 2, 3, 4) has a clearly expressed relaxational character. A slight diversity is manifest only in the absolute values of ε and tan δ in relation to the excess of Bi_2O_3 or TiO_2 in the systems n = 2 and n = 4. An important point is that the solid solutions of structural group a have a low conductivity, independent of field strength, whereas solid solutions of group b (samples of dark color) have a high conductivity that increases rapidly with increase of the field intensity (Fig. 6).

For materials of structural class 2 (the region of mechanical mixtures), an increase of bismuth titanate concentration produces not only a decrease of the magnitude of ε, but also a decrease of the frequency at which a decrease of ε is observed (Fig. 7); this suggests the possibility that a relaxational process is present in this concentration range as well. Further evidence of this is the temperature dependence of ε and tan δ at various frequencies.

For all the materials studied, no dielectric hysteresis loops were detected at temperatures above or below the maximum of ε.

The experimental results presented show that a high dielectric constant can be obtained in solid solutions by combining a favorable crystal structure with a relaxational polarization resulting from vacant sites in the lattice. These vacant sites can be obtained by formation of a solid solution of the second kind, of subtractive type, which is produced by replacement of some of the cations of the lattice with cations of higher valence. It should be mentioned that relaxational polarization can occur also in lattices of fluorite type, through formation of subtraction solid solutions ($ZrO_2 \cdot MgO$, $ZrO_2 \cdot CaO$); in this case, however, it does not, of course, lead to a high value of the dielectric constant, since the fluorite structure does not favor this.

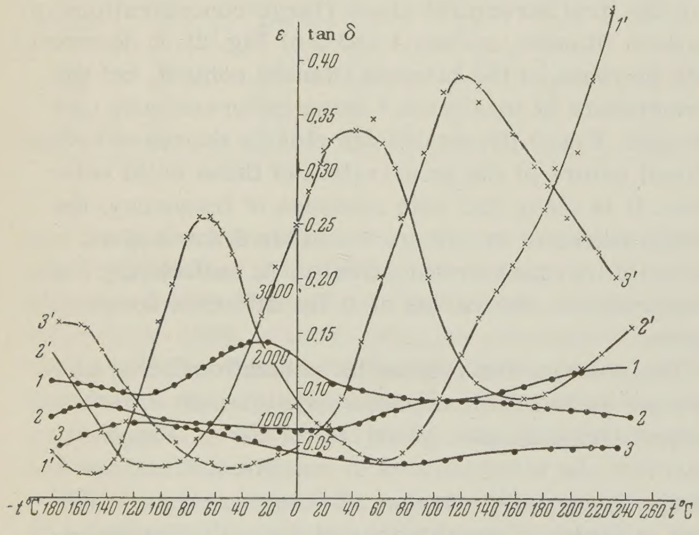

FIG. 4. Dependence of ε and tan δ on temperature for a dielectric of batch composition $SrTiO_3$ 77% and $Bi_2O_3 \cdot 2TiO_2$ 23% by weight, at various frequencies. 1 and 1′, ε and tan δ at frequency 10^3 cps; 2 and 2′, ε and tan δ at frequency 90 kcs; 3 and 3′, ε and tan δ at frequency 1.5 Mcs.

Our work has shown that the general method of obtaining dielectrics with a high dielectric constant, without ferroelectric properties, is applicable also to other compounds with the perovskite structure. Introduction of bismuth titanate into $BaTiO_3$ [$BaTiO_3 - Bi_2O_3 \cdot nTiO_2$] leads to a change in the nature of the polarization. With increase of the bismuth titanate concentration in this system to 15 mol percent, the spontaneous polarization gives way to a relaxational. At concentrations of less than 15 mol percent of 1/2 ($Bi_2O_3 \cdot 2TiO_2$), spontaneous polarization still occurs, but the Curie point moves to lower temperatures (curves 1 and 2 of Fig. 8). After the transition to relaxational polarization (bismuth titanate concentration greater than 15 mol percent), the peak in ε is washed out, and the temperature corresponding to this peak increases with increasing concentration (curves 4 and 5 of Fig. 8), just as in the system $SrTiO_3 - BiO_3 \cdot nTiO_2$. The presence of relaxational polarization is confirmed by the shift of the temperatures of maximum ϵ and tan δ with change of frequency (Fig. 9), which is not observed in ferroelectrics; also by the absence of dielectric hysteresis at all temperatures, and by the fact that ϵ is independent of the field intensity.

4. DISCUSSION OF EXPERIMENTAL RESULTS

It has been established experimentally that in the solid solutions $SrTiO_3 - Bi_2O_3 \cdot nTiO_2$ and $BaTiO_3 - Bi_2O_3 \cdot 2TiO_2$, at sufficient concentrations of bismuth titanate, a high dielectric constant is produced by a combination of favorable structure and of relaxational polarization, which occurs when vacant sites are present. We may suppose that the relaxational polarization originates from relatively large displacements of weakly bound ions, which by thermal motion surmount local potential barriers; for the bonds in the crystal lattices are predominantly of ionic nature. It is difficult to suppose that the large ions Ba^{2+}, Sr^{2+}, Bi^{3+}, or O^{2-} are capable of displacements through such a large distance, since they are densely packed in the lattices. The possibility that appears most plausible is that the large displacements are those of Ti^{4+} ions, whose dimensions are small. Formation of vacant sites in the strontium or barium parts of the lattice will lead to a shift of the equilibrium positions of the oxygen ions, i.e., to distortion of the oxygen octahedra that surround the titanium ions, and consequently to the appearance of several local minima of the potential energy of a titanium ion.

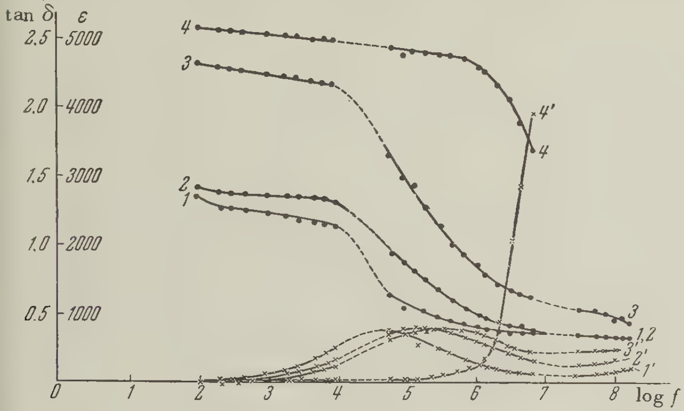

FIG. 5. Dependence of ϵ (curves 1 – 4) and tan δ (curves 1′ – 4′) on frequency (in cps) at room temperature, for dielectrics of the system $SrTiO_3 - Bi_2O_3 \cdot 2TiO_2$ with small bismuth titanate content. $SrTiO_3$ content (by weight): 1 and 1′, 77%; 2 and 2′, 79%; 3 and 3′, 85%; 4 and 4′, 93%.

At small concentrations of bismuth titanate (less than 15 mol percent) and at high sintering temperatures, solid solutions of substitutional type are formed, with liberation of oxygen and formation of quasi-free electrons (formula 1). With this, clearly, there also occur the experimentally observed change of color (darkening) of the specimens, which is linked with absorption in the visible part of the spectrum, and an increase of the electronic electrical conductivity. At the same time the dielectric constant and tan δ increase sharply, and the polarization, in the frequency and temperature range investigated, is of relaxational nature. This may be connected either with a heterogeneous microstructure of the polycrystalline dielectrics (the presence of phases of different conductivities must lead to relaxation of Maxwellian type) or with formation of F-centers with large orbital radii (a quasi-free electron bound by electrostatic forces, for example to a bismuth ion, will also be at a comparatively large distance from it). A unique answer to this question requires further investigations. In the case of solid solutions with vacant sites (SBT-1), it is very probable – as has been shown – that the relaxational polarization is ionic in nature; and analysis of the data by means of a simple model makes it possible to estimate a number of quantities characteristic of the ionic relaxation.

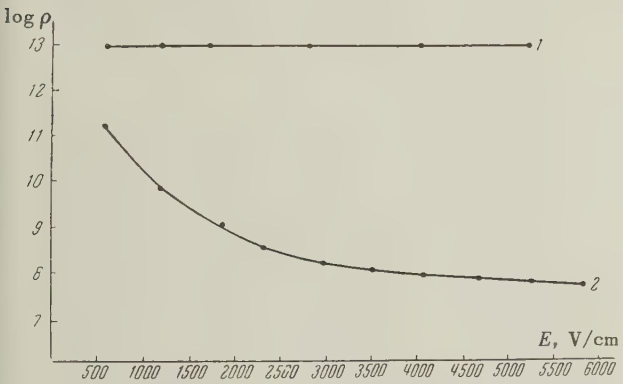

FIG. 6. Dependence of the logarithm of the specific resistance (in ohm cm) on the field intensity at room temperature, for strontium-bismuth titanates of batch composition (by weight): 1, $SrTiO_3$ 55%, $Bi_2O_3 \cdot 2TiO_2$ 45%; 2, $SrTiO_3$ 93%, $Bi_2O_3 \cdot 2TiO_2$ 7%.

As Landau remarked, it follows from simple phenomenological considerations that the relaxation time of the polarization is larger, the higher the static dielectric constant ϵ_0.* In fact, if we expand the mean

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*We shall show shortly that any molecular theory of relaxational polarization also gives, in general, this same result.

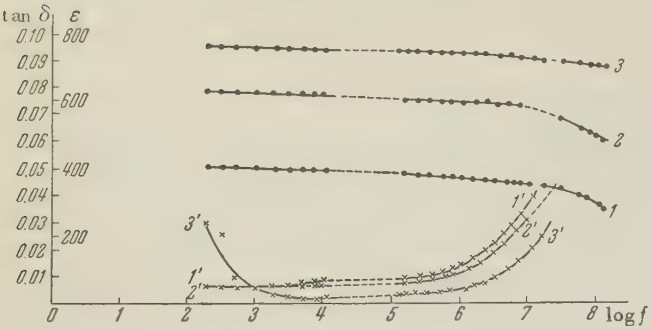

FIG. 7. Dependence of ϵ (curves 1 — 3) and tan δ (curves 1′ — 3′) of frequency (in cps) at room temperature, for strontium-bismuth titanates. 1 and 1′, batch composition (by weight) $SrTiO_3$ 26%, $Bi_2O_3 \cdot 2TiO_2$ 74%;) 2 and 2′, $SrTiO_3$ 35%, $Bi_2O_3 \cdot 2TiO_2$ 65%; 3 and 3′, $SrTiO_3$ 55%, $Bi_2O_3 \cdot 2TiO_2$ 45%.

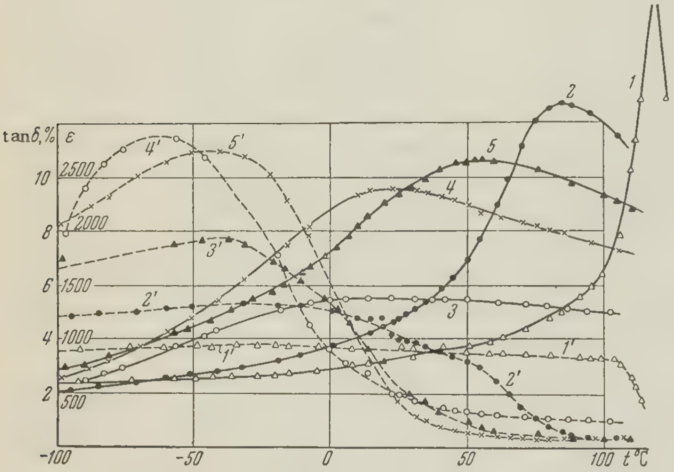

FIG. 8. Dependence of ϵ and tan δ on temperature at frequency 1000 cps, for dielectrics of the system $BaTiO_3 - BiTiO_{7/2}$: 1, 0.95 $BaTiO_3$ + 0.05 $BiTiO_{7/2}$; 2, 0.9 $BaTiO_3$ + 0.1 $BiTiO_{7/2}$; 3, 0.85 $BaTiO_3$ + 0.15 $BiTiO_{7/2}$; 4, 0.7 $BaTiO_3$ + 0.3 $BiTiO_{7/2}$.

field intensity as a series in the frequency,

$$E_{\text{av}} = \frac{4\pi I}{\varepsilon_0 - 1} + 4\pi\alpha' \dot{I} + 4\pi\beta' \ddot{I} + \dots \qquad (5')$$

(I is the polarization, or electric moment of unit volume), and retain only three terms of this series, we get

$$\varepsilon^* = 1 + (\varepsilon_0 - 1)\,[1 + i\omega(\varepsilon_0 - 1)\,\alpha' - \beta'(\varepsilon_0 - 1)\,\omega^2]^{-1},$$

$$\varepsilon' = 1 + (\varepsilon_0 - 1)\,[1 - \beta'(\varepsilon_0 - 1)\,\omega^2]\,\{[1 - \beta'(\varepsilon_0 - 1)\,\omega^2]^2 + \omega^2(\varepsilon_0 - 1)^2\,\alpha'^2\}^{-1}, \qquad (6)$$

where $(\epsilon_0 - 1)\,\alpha' = \theta$ is the relaxation time of the polarization.

In this case, $\epsilon' \rightarrow 1$ at high frequencies. However, a great quantity of experimental data for polar liquids and for crystals that contain dipolar groups[3] shows that the dielectric constant diminishes sharply as the frequency increases over a very limited interval, and remains practically independent of frequency outside that interval. At sufficiently low frequencies ϵ' is close to ϵ_0; at sufficiently high frequencies, $\epsilon' \rightarrow \epsilon_\infty > 1$.

As has been shown, in the solid solutions SBT-1 a partial replacement of strontium ions by bismuth ions not only increases the dielectric constant, but sharply changes the nature of its dependence on temperature and frequency. For sufficiently high frequencies, $\epsilon_\infty \sim 250$; but $\epsilon_0 > 1000$. It is clear that simple phenomenological considerations [formula (6)] are not sufficient for the interpretation of the relations that have been established.

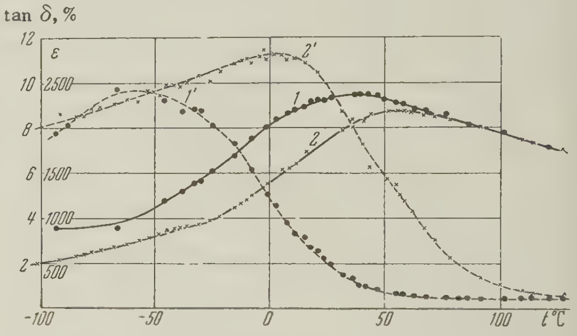

FIG. 9. Dependence of ϵ and tan δ on temperature for a dielectric of batch composition 0.8 $BaTiO_3$ + 0.2 $BiTiO_{7/2}$; 1, f = 10^3 cps; 2, f = 5×10^5 cps.

In this connection, "model" interpretations of the nature of the relaxational polarization in SBT-1 acquire some additional basis; the more so, because it was considerations of precisely this type that led to the creation of these new dielectrics. The starting point here is the separation of the polarization into two parts: (a) a polarization (ionic, electronic) that becomes established rapidly, and (b) a polarization that becomes established comparatively slowly and that depends comparatively strongly on temperature — the so-called relaxational polarization, which (like the first part) is proportional to the field; for the ϵ' of the dielectrics SBT-1 is independent of the field intensity even at strong fields. In contrast to the polarization of polar liquids and of crystals containing dipolar groups, the first part of the polarization is due not only to electronic displacements but also to ordinary elastic displacements of ions; and the relaxational polarization is due not to orientation of dipolar groups, but to passage of some of the ions across local potential barriers.

It must be stressed that such a separation of the complete polarization into a part that is established in a short time and a part that is established slowly, regardless of the mechanism of polarization, always leads to a formula of Debye type for the dependence ϵ' and ϵ'' (and consequently also of tan δ) on frequency and temperature. Since the effective field for dense phases is not equal to the macroscopic mean field, the relaxation time of the slowly established polarization is always approximately proportional to the static dielectric constant; this is in agreement with the simple phenomenological considerations presented above. Such an approach to the treatment of experimental data, in contrast to the phenomenological, provides a possibility of giving the coefficients in the series (5′) a descriptive interpretation and of determining to what degree one or another mechanism of relaxational polarization agrees with experimental data. The "model" method of treating experimental data usually encounters difficulty with regard to the calculation of the deviation of the effective (local) field from the mean field. However, when the experimental data are well described by a formula of Debye type, as is true in the case under consideration, it is possible to circumvent this difficulty by introducing the so-called mean or generalized internal-field coefficients β_1 and β_2.[1,3]

It should be mentioned that formulas of Debye type coincide with formulas (6), obtained from the series (5′), if in the latter we keep only the first two terms instead of three, and if we set ϵ_∞ equal to unity. Then the polarization relaxation time is $\theta = (\epsilon_0 - 1)\,\alpha'$. As is well known, formulas of Debye type always give temperature-frquency maxima of tan δ, and also a temperature maximum of ϵ' at given frequency; as was pointed out above, these are observed in all the new dielectrics obtained and investigated in this research. It is noteworthy that the frequency shifts of the curves of temperature variation of ϵ' and tan δ of the solid solutions SBT-1, as observed experimentally, correspond to a large diminution of polarization relaxation time θ with increase of temperature, since the curves move toward higher temperatures upon increase of frequency. However, this diminution of θ can not be attributed solely to decrease of ϵ_0. For example, upon increase of the temperature by 100°, the frequency corresponding to the temperature maxima and tan δ and ϵ' increases by four orders of magnitude. Thus it follows from the experimental data that the coefficient α' in the series (5′) decreases rapidly (almost exponentially) with temperature. In a model theory this corresponds to an exponential decrease with temperature of the relaxation time τ of weakly bound ions, which is proportional to θ; the coefficient of proportionality depends on ϵ_0 and ϵ_∞. If we use the generalized internal field coefficients,

$$\theta = \frac{\varepsilon_0 - 1 + 4\pi/\beta_1}{\varepsilon_\infty - 1 + 4\pi/\beta_1}\,\tau; \quad \tau = Ce^{U/kT}, \tag{7}$$

Here U is the activation energy; β_1 is the coefficient of proportionality between the electric moment of unit volume produced by relaxation polarization alone and its contribution to the mean local (internal) field; C is a quantity of the order of the reciprocal of the frequency of oscillation of the weakly bound ions about an equilibrium position and is only slightly dependent on temperature. By comparing the conditions for a temperature maximum of tan δ and of ϵ' at frequency ω, or by considering the shift of the temperature maximum of tan δ with change of frequency, it is possible to calculate the value of the activation energy of the weakly bound ions responsible for the relaxational polarization, by the following formulas.

In the first case

$$U \approx \frac{kT_1 T_2}{T_1 - T_2} \ln\left[\frac{\varepsilon'_{\max} - \varepsilon_\infty}{\varepsilon''_1}\,\frac{T_1}{T_2}\sqrt{\frac{\varepsilon_{02}}{\varepsilon_\infty}}\right], \tag{8}$$

under the condition that $(T_1/T_2) - 1 \ll (\epsilon'_{\max} - \epsilon_\infty)^2 kT_1^2/2\epsilon_1''^2\, UT_2$, where T_1 and T_2 are the temperatures of maximum ϵ' and of maximum tan δ; $\epsilon''_1 = \epsilon'_{\max} \tan \delta_1$ at temperature T_1; and ϵ_{02} is the static dielectric constant at temperature T_2.

In the second case

$$U = \frac{kT_2^* T_2^{**}}{T_2^{**} - T_2^*} \ln\left[\frac{\omega^{**} T_2^* D^{*2}}{\omega^* T_2^{**} D^{**2}}\sqrt{\frac{\varepsilon_{0T_2^*}}{\varepsilon_{0T_2^{**}}}}\right], \tag{9}$$

where $D = (\epsilon'_{\max} - \epsilon_\infty)/\epsilon''_1$; the symbols * and ** refer to frequencies ω^* and ω^{**}.

Table III gives the original experimental data for one of the solid solutions of group a (SBT-1, Fig. 3) and the values of the activation energy of the weakly bound ions, as obtained from (8) with $\epsilon_\infty = 250$.

TABLE III

f, cps	T_1 °K	T_2 °K	$\varepsilon'_{\max}$	ε_{01}	ε_{02}	tan δ_1	ε''_1	U, ev
400	173	128	1310	1330	1450	0.033	43.2	0.19
1000	178	133	1290	1310	1420	0.030	38.7	0.20
5000	183	143	1260	1300	1400	0.038	47.8	0.23
$2.5 \cdot 10^5$	200	156	1110	1220	1380	0.037	41.1	0.26

The values of ϵ_{01} and ϵ_{02} given in Table III were obtained by extrapolation to temperatures T_1 and T_2 of the straight line that describes the dependence of ϵ' on temperature for $T > 273$°K. The values of activation energy calculated from (8) for different frequencies are close, but they increase slightly with increase of frequency. The values of U found with formula (9) from the experimental data of Table III, for various pairs of frequencies, lie within the limits 0.34 to 0.55 ev and likewise increase of frequency. The appreciable scattering of the calculated values of U, their increase with frequency, and also the increase of tan $\delta_{\max}$ with increase of frequency (Fig. 3), may be attributed to the presence of two or more sets of potential barriers and relaxation times; these were not taken into account in the calculations. The mean value of U depends comparatively slightly on the composition of the solid solutions SBT-1. However, upon increase of the bismuth titanate concentration the relaxation time at given temperature and consequently the activation energy increase slightly (Fig. 2). From knowledge of U it is possible to estimate the internal-field coefficient β_1 from the approximate relation

$$\left(\frac{\varepsilon'_{\max} - \varepsilon_\infty}{\varepsilon''_1}\right)^2 \approx \frac{2U}{kT_1} \cdot \frac{\beta_1(\varepsilon_\infty - 1) + 4\pi}{\beta_1(\varepsilon_{01} - 1) + 4\pi}. \tag{10}$$

It is found that for the solid solutions SBT-1, β_1 has a very small value ($\beta_1 \to 0$ or $\beta_1 = -10^{-2}$). The relaxation time of the weakly bound ions at temperature T_1 can be estimated, for example, by means of the following approximate formula (for $U \gg kT_1$):

$$\omega\tau_1 \approx kT_1(\varepsilon'_{\max} - \varepsilon_\infty)/2U\varepsilon''_1. \tag{11}$$

The calculations give: for $T_1 = 173$°K, $\tau = 2.4 \times 10^{-4}$ sec; for $T_1 = 200$°K, $\tau_1 = 0.4 \times 10^{-6}$ sec; the coefficient of the exponential in the expression $\tau = C \exp(U/kT)$ is 10^{-12} to 10^{-13} sec. At room temperature ($T = 300$°K), τ is of order 10^{-8} sec; therefore in a frequency range near 10^8 cps, there should occur a diminution of ϵ' and a frequency maximum of tan δ, in agreement with experimental data. The magnitude of the relaxation polarization is determined[1,3] by the expression $n'q^2x^2/12kT$, where n' is the number of weakly bound ions per cm^3 of the dielectric, q is the ionic charge and x is the distance between neighboring local minima of potential energy of a weakly bound ion. It is easy to see that the following relation holds:

$$\frac{n' q^2 x^2}{12kT} = \frac{4\pi(\varepsilon_0 - \varepsilon_\infty)}{[4\pi + \beta_1(\varepsilon_{01} - 1)][4\pi + \beta_2(\varepsilon_\infty - 1)]}, \tag{12}$$

where β_2 is the coefficient of proportionality between the ordinary elastic polarization (due to ionic and electronic displacement) and its contribution to the mean local field. The coefficient β_2 can be estimated by means of the well-known structural internal-field coefficients for a perovskite-type lattice,[3,4] if one assumes that the presence of lattice defects does not appreciably change the internal field produced by ordinary elastic polarization. Such a calculation, with consideration only of the polarization due to electronic displacement, gives $\beta_2 \approx 0.9(4\pi/3)$. An estimate of β_2 can be made by another method, as was done in Ref. 1; according to such an estimate, $\beta_2 \approx 5$. Finally, β_2 can also be found by drawing on additional experimental data. However, the precision of the last estimate is still poorer than that of the first two. On setting $\beta_1 = 0$ and $\beta_2 = 0.9(4\pi/3)$, we get (for $T_1 = 173$°, $\epsilon_{01} = 1330$, $\epsilon_\infty = 250$) $n'q^2x^2/12kT \approx 1$. If we furthermore assume that the number of weakly bound ions is close to the number of bismuth ions in the lattice of the solid solution SBT-1, then for 20 mol percent concentration of bismuth titanate, $n' \sim 9 \times 10^{20}$ cm^{-3}. There is reason to suppose that the weakly bound ions are the titanium ions in the distorted oxygen octahedra (cf. above); then $q = 4 \times 4.8 \times 10^{-10}$ cgs units. Hence for $T = 173$° we find that the relaxational displacement of the titanium ions is $x \sim 0.3 \times 10^{-8}$ cm. Passage of the titanium ions over such a comparatively short distance within the oxygen octahedron corresponds, as was to be expected, to a small value of the activation energy, ~ 0.3 ev.

Within the framework of this hypothesis of a transition of weakly bound ions across a potential barrier, it is easy to explain why linearity of the polarization persists even at high fields. The linearity can fail if the intensity of the field acting on the weakly bound ion is of order $E_{cr} \approx 2kT/qx$, since then the work by the field in a displacement x is comparable with the energy of thermal motion. For $q = 4 \times 4.8 \times 10^{-10}$, $x = 0.3 \times 10^{-8}$ cm, and T = 300°K, $E_{cr} = 2 \times 10^6$ volt/cm. Upon going over from local to mean field, on the assumption that $\beta_1 \approx 0$ and $\beta_2 \approx 4\pi/3$, we get $E_{av\,cr} \approx 3\,E_{cr}/(\epsilon_\infty + 2) = 2 \times 10^4$ v/cm, whereas the mean breakdown intensity is about 2×10^5 v/cm, i.e., a departure from linearity must take place only at fields close to the breakdown field. Thus the hypothesis of a transition of weakly bound ions across potential barriers in the disturbed perovskite lattice is in more or less satisfactory agreement with experimental data. Other models fit the experiment less satisfactorily. For example, an anharmonic oscillator model must lead to nonlinear polarization even in weak fields. The hypothesis of resonance of the field oscillations with the optical lattice oscillations, which according to formula (6) should occur at lowered frequencies $\omega \approx \omega_0/\epsilon_0{}^{1/2}$ (dispersion in barium titanate[5]), does not agree with experiment in the present case, in view of the change of the frequency of maximum tan δ upon comparatively slight change of temperature. Further studies of the polarization of SBT-1 should lead to the determination of the distribution function of the relaxation times, and also to a clarification of the temperature dependence of the static dielectric constant ϵ_0 in the low-temperature range; and, in particular, to a clarification of the occurrence of "orientational melting," which occurs in crystals containing dipolar groups.

The very high dielectric constant of the SBT-2 dielectrics, as has already been pointed out, is also due to relaxational polarization, which however, is of different character. The curves of temperature dependence of ϵ' and tan δ have a more complicated character (Fig. 4) than do those for the SBT-1 dielectrics. However, an estimation of the activation energy from the frequency shift of the temperature maximum of tan δ (Fig. 4) likewise gives a small value, 0.2 to 0.4 ev. The nature of the polarization of these dielectrics is subject to further study.

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SOVIET PHYSICS JETP VOLUME 6(33) NUMBER 2 FEBRUARY, 1958

THE ELASTIC SCATTERING OF 307 MEV π^--MESONS FROM HYDROGEN

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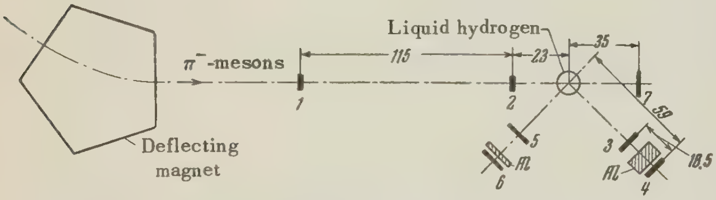
Scintillation counters were used to study the angular distribution of π^--mesons scattered elastically from hydrogen. The energy of the incident π^--mesons was 307 ± 9 Mev. The angular distribution obtained is closely represented by $d\sigma/d\omega = (0.56 + 0.42 \cos\vartheta + 1.1 \cos^2\vartheta) \times 10^{-27}$ cm^2 sterad^{-1} in the center of mass system. A phase-shift analysis was carried out, using only S and P waves. Comparison of the data with the dispersion relations leads to the conclusion that the coupling constant f^2 for the meson-nucleon interaction is about 0.08.

TILL recently, the angular distribution of π^--mesons scattered elastically from hydrogen has been studied in detail only at energies below 220 Mev. The data on this interaction at ~ 300 Mev obtained with π^--mesons from the synchrocyclotron of the Joint Institute for Nuclear Research[1,2] have low statistical accuracy. In the present work, the angular distribution of π^--mesons from the process $\pi^- + p \rightarrow \pi^- + p$ was measured using scintillation counters.

Mesons were obtained from a beryllium target inside the synchrocyclotron of the Joint Institute for Nuclear Research. Upon passing through a collimator in the magnet yoke, they were bent by a focusing magnet and fell on a hydrogen target. The energy of the π^--mesons was 307 ± 9 Mev, and the beam intensity about 100 mesons/cm^2 sec.

The geometry of the experiment is shown in the figure. The beam incident on the hydrogen target was detected by counters 1 and 2, 6 × 6 cm in size and connected in coincidence. The scattered mesons were counted at two angles simultaneously by the two telescopes, each of which consisted of two liquid scintillation counters (3, 4 and 5, 6) 12.6 × 11.5 cm in size. The pulses from the photomultipliers were first shaped and then transmitted on high-frequency cables to a multi-channel coincidence circuit with a resolving time of ~ 1.2×10^{-8} sec. This circuit used germanium detectors and had a sensitivity ~ 0.5 volts. The number of double coincidences D from detectors 1 and 2 were registered on a scaler with resolving time ~ 10^{-7} sec. Four-fold coincidence Q from the counters 1, 2, 3, 4 and 1, 2, 5, 6, were passed through a gate operated by counter 7, which was in anti-coincidence with counters 1 and 2, and then registered mechanically. The anti-coincidence counter 7 was necessary to reduce the background of accidental coincidences. The anti-coincidence circuit used germanium detectors and had a resolving time of ~ 2.2×10^{-8} sec.



Geometry of the experiment. 1, 2 — scintillation counters, 6 × 6; 3, 4, 5, 6 — scintillation counters, 12.6 × 11.5; 7 — anti-coincidence counters, 12.6 × 11.5 cm.

The target was liquid hydrogen in a polystyrene foam bottle. There was about 0.735 g/cm^2 of hydrogen in the path of the beam, while the amount of material in the target was 0.35 g/cm^2. Measurements were made at angles of 30, 45, 60, 80, 100, 125, and 152 degrees in the laboratory frame of reference. During the measurements at 30, 45, and 60 degrees, counters 4 and 6 were shielded by 28.4, 20.2, and 9.4 g/cm^2 respectively of aluminum to absorb recoil protons. At the other angles, counters 4 and 6 were shielded by 5.4 g/cm^2 of aluminum to absorb protons from stars.

The differential cross section was computed from the formula

$$\left(\frac{d\sigma}{d\omega}\right)_{\text{lab}} = \frac{(Q/D)_{\text{with H}} - (Q/D)_{\text{without H}}}{kN\omega}$$

where $N = 0.443 \times 10^{24}$ is the number of hydrogen atoms per cm², and ω is the solid angle defined by the telescopes. k is a correction factor which takes account of the decrease in the telescope efficiency due to the aluminum filters, the increase in counting rate due to the conversion in counters 3 and 5 of γ-rays from π^0-decay, μ-meson contamination of the beam, meson absorption in the target walls and in counters 3 and 5, counting losses at high counting rates, and a number of other small corrections.

Measurements to be reported on spearately showed that at π^--meson energies of 307 Mev, meson production by mesons is appreciable. At an angle of 80° in the laboratory frame, the number of mesons produced in the hydrogen is about 7% of the number of elastically scattered π^--mesons. In correcting for this effect at other angles, the following assumptions were made: in the center-of-mass system, all the kinetic energy is carried away by the π-mesons, the mesons have a triagnular energy spectrum, and their angular distribution is isotropic.

TABLE I. Angular distribution of π^--mesons elastically scattered from hydrogen, $E_{\pi^-} = 307 \pm 9$ Mev.

Angle, c.m.s.	Differential cross section in 10^{-27} cm²/sterad.	Angle, c.m.s.	Differential cross section in 10^{-27} cm²/sterad.
41°20′	1.30 ± 0.27	118°59′	0.61 ± 0.07
60°35′	1.05 ± 0.13	140°01′	0.89 ± 0.1
78°28′	0.75 ± 0.09	160°16′	1.12 ± 0.12
99°57′	0.49 ± 0.06	—	—

Table I shows the values obtained for the differential cross section in the center-of-mass system after all corrections have been made. The errors shown are standard deviations, except for that at 41°. The results of repeated measurements at 41° differed among each other by more than the statistical error. The error quoted for this angle is the mean square error of a single measurement.

A least squares fit of the data to the expression

$$(d\sigma / d\omega)_{\text{c.m.s.}} = a + b \cos \vartheta + c \cos^2 \vartheta \tag{1}$$

leads to the following values for the coefficients (in 10^{-27} cm² sterad^{-1}):

$$a = 0.56 \pm 0.05; \; b = 0.42 \pm 0.11; \; c = 1.1 \pm 0.16. \tag{2}$$

Upon integrating the expression (1), the integral cross section for the process $\pi^- + p \rightarrow \pi^- + p$, $\sigma_{\pi^- \rightarrow \pi^-}$, is found to be $(11.3 \pm 0.9) \times 10^{-27}$ cm². In Refs. 1 and 2 this cross section was found to be $(11 \pm 2) \times 10^{-27}$ cm² and $(11 \pm 4) \times 10^{-27}$ cm² respectively.

TABLE II.
Phase Shifts

α_1	α_{11}	α_{13}	M
5°	8°	—8.2°	1.4
1.5°	—22°	18°	1.5
15°	—19.5°	19.5°	1.3
11.8°	35.5°	7°	1.3

A preliminary phase-shift analysis has been carried out on the basis of the following simplifying assumptions: (1) only S and P waves need be considered and (2) the relation between the phase shifts and the angular distribution of elastically scattered π^--mesons at ~ 300 Mev is the same as it would be if there were no inelastic processes. The phase shifts describing the interaction in S, $P_{1/2}$, and $P_{3/2}$ states with isotopic spin $3/2$ are denoted by α_3, α_{31}, and α_{33}. These phase shifts were taken from Ref. 3. They are $\alpha_3 = -23.2°$, $\alpha_{31} = -8.4°$, and $\alpha_{33} = 133.2°$. The phase shifts describing the interaction in S, $P_{1/2}$, and $P_{3/2}$ states with isotopic spin $1/2$, are denoted by α_1, α_{11}, and α_{13}. Four sets of phase shifts were found, each minimizing the expression

$$M = \sum_{1}^{7} (\Delta_i / \varepsilon_i)^2,$$

where Δ_i is the difference between the experimentally observed differential cross section at a given angle and the value computed from the phase shifts, while ϵ_i is the experimental error in the measured differential cross section. The phase shifts and the corresponding value of M are shown in Table II.

It should be noted that although these sets of phase shifts all describe the angular distribution of elastically scattered π^--mesons equally well (M ~ 1.5, with an expected value of 4), none of them describe the angular distribution of π^0-mesons adequately. (The latter has been obtained in preliminary experiments.) The reason for this may appear after some supplementary measurements on exchange of scattering have been carried out, and a more detailed phase shift analysis, taking D waves into account, has been made. From a theoretical point of view, it is also not clear whether we are justified in neglecting the inelastic processes or not.

Measurements on the elastic scattering of π^--mesons can be used to obtain information on the coupling constant f between mesons and nucleons. It is well known[4] that dispersion relations give the energy dependence of the real part $D_-(0)$ of the π^--meson scattering amplitude at 0°. The curve depends on the value of f^2. Puppi and Stanghellini[5] pointed out that the experimental data for π^--mesons with energies less than 180 Mev leads to $f^2 \sim 0.04$, while the data for π^--meson energies ~ 220 Mev lead to $f^2 \sim 0.08$. The experiments on the interaction of π^+-mesons with protons at energies up to 400 Mev lead to $f^2 \sim 0.08 - 0.1$. The relation between $D_-(0)$ and the coefficients in (1) is

$$|D_-(0)| = \sqrt{(a+b+c) - (k\sigma_t/4\pi)^2},$$

where σ_t is the total cross section for the interaction of π^--mesons with hydrogen,* and k is the wave number of the π^--meson.

The value obtained in this way for $D_-(0)$ is $D_-(0) = (-0.24 \pm 0.05) \times 10^{-13}$ cm at 307 Mev and is in satisfactory agreement with the curve corresponding to $f^2 = 0.08$.[5,6] This confirms the contradiction found by Puppi and Stanghellini. As these authors assume, the discrepancy with the results using π^--mesons with energies less than 180 Mev can be explained by electromagnetic effects and the $\pi^\pm - \pi^0$ mass difference.

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*The value $\sigma_t = (31.6 \pm 1.6) \times 10^{-27}$ cm^2 was obtained by interpolating the data of Ref. 7.

SOVIET PHYSICS JETP VOLUME 6(33) NUMBER 2 FEBRUARY, 1958

YIELD OF CALIFORNIUM ISOTOPES PRODUCED IN REACTIONS BETWEEN CARBON IONS AND URANIUM NUCLEI

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The yield of the californium isotopes Cf^{264}, Cf^{245} and Cf^{244} produced in the reaction between 90 Mev carbon ions and a thick uranium target was investigated. The absolute californium yield per incident particle is 1.5×10^{-9} Cf^{246} nuclei, 3.0×10^{-9} Cf^{245} nuclei and 9×10^{-11} Cf^{244} nuclei. Comparison of the obtained results with the cross-section for fission of uranium by carbon ions indicates that fission is 3.8×10^3 times more probable in this case than neutron evaporation from the compound Cf^{250} nucleus.

IN the production of transuranic elements by the reaction of multiply charged ions with active elements there is natural interest in data on the yield of nuclei resulting from the complete absorption of the incoming particle by the target nucleus followed by emission of several neutrons. The first experiments on such reactions were carried by Ghiorso et al.[1] in work on the production of californium in irradiations of uranium with carbon ions. More detailed investigations of the reactions of carbon ions with uranium were carried out by Fremlin et al.[2] In both cases carbon ions having a large spread in energy were used. It appeared interesting to investigate such reactions using projectiles having a more narrowly defined spectrum of energies.

In the present work there were determined the yields of isotopes of californium produced in the bombardment of uranium by monoergic carbon ions. The cyclotron accelerated quadruply charged carbon ions, which started at a slit-type source, and had an energy of 90 Mev at a radius of 67 cm. The maximum ion current at this radius was ~ 1μa. The target consisted of metallic uranium sheets having dimensions of 15 × 7 mm^2 and a thickness of 30μ; they were clamped on a water cooled probe. To avoid possible destruction of the target, carbon-ion currents of ~ 0.1μa were used. The beam current measurement was made by recording the ions passing through the target and impinging on a collector, using a conventional electrometric setup.

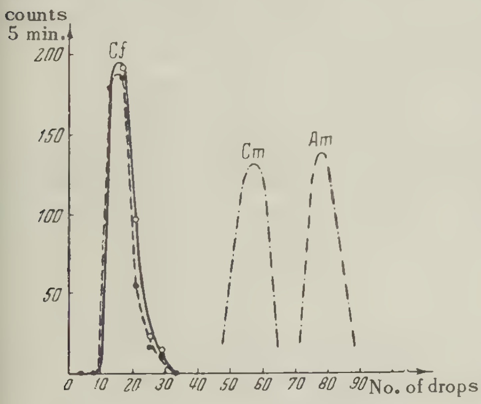

FIG. 1. Elution curves from the chromatographis separation. Solid dots (dashed curve) — Cf^{246}, open points (solid curve) — Cf^{245}.

In calculating the impinging beam current, account was taken of the fact that quadruply charged carbon ions change their effective charge to approximately six elementary charges in passing through a 30μ thick target foil. The irradiation time in different experiments was between 0.5 and 2.5 hours.

The bombarded foil was dissolved in concentrated nitric acid and the actinide fraction was separated from the uranium matrix by precipitation with lanthanum fluoride. The separation of the actinides was carried out on a cation column containing Dowex-50 X-12 at a temperature of 87°C. The complexing solution used consisted of a 0.4 M solution of α-hydroxyisobutyric acid at a pH = 4.1. The chromatographic elution curve of the actinides is presented in Fig. 1. This figure also shows the positions of the maxima of Am and Cm obtained under identical conditions.

The measurement of the energies of the α-particles and the half-lives of the α-active isotopes were carried out using an ionization chamber with a spherical electrode and a 20 channel pulse analyzer.

In the californium fraction there were identified, by means of the energies of the α-particles and the half-lives, the following isotopes of californium: Cf^{246} and Cf^{245}.[3,4] Figure 2 shows the α-particle spectrum of the californium fraction at different times. The decay curves of the Cf^{246} and Cf^{245} are presented in Figs. 3 and 4.

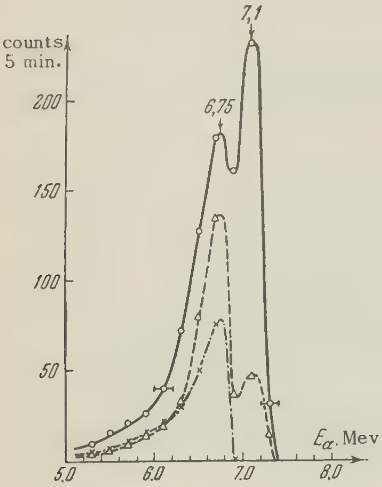

FIG. 2. Spectrum of α-particles in the californium fraction at various times. (Solid curve) — 1 hr. 33 min. after the end of bombardment; (dashed curve) — 3 hrs. 23 min. after the end of bombardment; (dot-dashed curve) — 24 hrs. 49 min. after the end of bombardment.

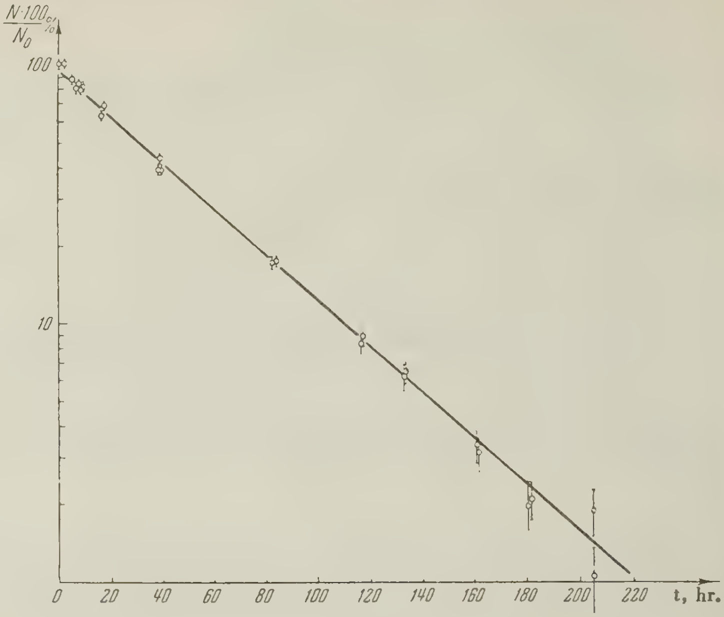

FIG. 3. Decay curve of Cf^{246}. $T_{1/2} = 35$ hrs.

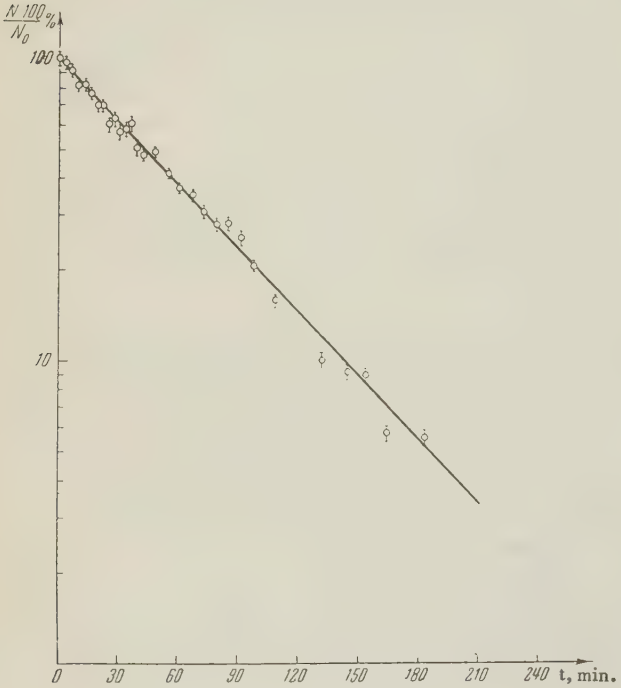

FIG. 4. Decay curve of Cf^{245}. $T_{1/2} = 44$ min.

The 25-min Cf^{244} was not determined in these experiments. One bombardment, in which the α activity of the irradiated metallic uranium was measured directly 5 min after the end of the irradiation, failed likewise to give positive results. An evaluation of the yield of Cf^{244} from the content in the curium fraction of its daughter Cm^{240} formed before the separation of the actinides was not unique since the isotope Cm^{240} can be produced directly by the reaction U(C, α6n).

The results obtained give the yield of α active nuclei of the various isotopes of californium formed in the bombardment of a thick uranium target by 90 Mev carbon ions according to reactions of the type U(C, xn) Cf. The relative yields of these α-active nuclei are presented below. The yield of the α-active isotope Cf^{245} is taken as unity. An upper limit is given for the yield of the isotope Cf^{244}.

Isotope	Cf^{246}	Cf^{245}	Cf^{244}
Production reaction Yield of α-active nuclei	(C, 4 *n*) 1.65±0.15	(C, 5*n*) 1.00	(C, 6*n*) ≤0.1

The absolute yields of these californium isotopes per incident particle are, 1.5×10^{-9} for Cf^{246}, $\sim 3.0 \times 10^{-9}$ for Cf^{245} and $\leq 9 \times 10^{-11}$ for Cf^{244}. The total yield of these isotopes is equal to $\sim 4.5 \times 10^{-9}$ nuclei per incident carbon particle. In the calculations of the absolute yield of the californium isotopes account was taken of the fact that the fraction of the nuclei decaying by α emission is 100% for Cf^{246} and Cf^{244} (3) (5) and ~ 30% for Cf^{245}. (5) The yield of californium through all chemical operations and chromatographic separations was taken as 50% on the basis of data on the yields of Am and Cm in identical experiments. The number of californium atoms separated in each experiment was ~ 3×10^5 of Cf^{246} and ~ 1×10^5 of Cf^{245}.

Since the yield of californium isotopes is determined by the probability of formation of the compound nucleus Cf^{250} and subsequent competition between neutron emission and fission, it appeared interesting to compare the probabilities of these two last processes. In order to evaluate the probability of fission of californium nuclei there were utilized unpublished data on the dependence of the fission cross-section of uranium, σ_f, by carbon ions as a function of the energy of the particles. These data were obtained by V. A. Druinyi, S. M. Polikanov, and G. N. Flerov:

E_C, (Mev)	90	80	75
σ_f, (cm^2)	$\sim 6 \times 10^{-25}$	$\sim 2 \times 11^{-25}$	$\sim 1 \times 10^{-25}$

From these numbers and use of the range energy relation for carbon ions in uranium it was established that every carbon particle of 90 Mev produces 1.7×10^{-5} fissions in passing through a thick target of uranium. Comparison of this figure with the yield of califorium nuclei leads to the conclusion that the fission processes in 3.8×10^3 times more probable than the process of neutron evaporation from the compound nucleus in the bombardment of a thick uranium target by 90 Mev carbon ions. This value for the ratio of probabilities of fission to those of californium isotope production is significantly different from the value (~ 10^5) obtained in published work.[5]

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DETERMINATION OF THE RANGE-ENERGY RELATION FOR NITROGEN AND OXYGEN IONS IN PHOTOGRAPHIC EMULSIONS

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The dependence of range on energy has been measured for nitrogen and oxygen ions in Ilford E-1 emulsions for the energy interval 3 — 120 Mev. The results are compared with theoretical predictions. The energy at which the nitrogen and oxygen ions lose all their electrons traversing the matter has been estimated.

THE experimental determination of the range-energy dependence for N and O ions is of interest because theoretical estimates meet with considerable difficulties. The phenomena of electron loss and capture in traversing the matter manifest themselves for these ions in a wide range of energies. The Bethe-Bloch formula, which does not account for charge variation, cannot therefore be applied in this case.

Several authors calculated the range-energy relation in photographic emulsions for ions with $3 \leq Z \leq 10$.[1-3] These calculations, involved semi-empirical constants connecting the probability of electron loss and capture with the ion velocity. A few experimental points were obtained for C ions by Miller[4] and for N ions by Chaminade et al.[5] Reynolds et al. measured the range of N ions in the emulsion[6] and in nickel[7] in the energy interval 5 — 30 Mev.

We carried out an investigation of the range-energy dependence for the N and O ions in emulsion in the energy range 3 — 120 Mev, using for this purpose a beam of ions accelerated by a 150 cm cyclotron. The N and O ions were accelerated to large energies by the multiple frequency method in which the ions are initially accelerated while doubly charged and then are stripped on the atoms of the residual gas and as sextuply ionized are accelerated to high energies. Since the process of electron loss takes place in the whole volume of the cyclotron chamber, ions with different energies, up to a maximum of 130 Mev are present at the final orbit. No special measures for beam ejection were used but a part of the ions scattered by the dee walls found its way into the ejection system.

The beam passed through a magnetic analyzer in order to obtain monochromatic lines. An aluminum foil 6 μ thick in which high energy ions lost their electrons completely and low-energy ions attained the equilibrium charge was placed before the analyzer. The foil had also as its purpose to prevent doubly-charged ions from reaching the photographic plate. The plates were placed in a vacuum chamber, at the focal point of the analyzer. The magnetic field of the analyzer was calibrated by means of protons and molecular hydrogen. The proton energy was measured by N. D. Fedorov. Accurate test experiments demonstrated that in the region in which the measurements were carried out the magnetic field depends linearly on the current in the magnet coil. The ion energy, corresponding to a certain value of the current I_i could, therefore, be found from the relation

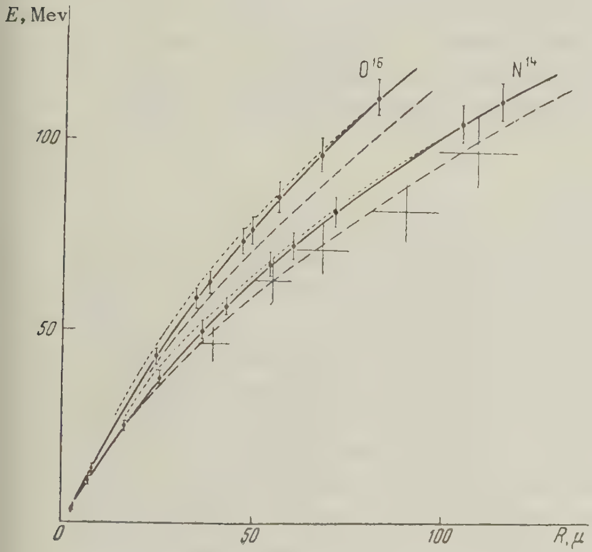
$$E_i = \frac{M_H}{M_i}\left(\frac{Z_i}{Z_H}\right)^2\left(\frac{I_i}{I_H}\right)^2 E_H,$$

where Z_i, M_i are the charge and the mass of the ion, Z_H and M_H are the charge and mass of molecular hydrogen, I_H is the coil current for which molecular hydrogen of a known energy passes through the analyzer.

Varying the magnetic field of the analyzer it was possible to obtain on the photographic plate N and O ions of different energies up to 130 Mev. Besides the group of ions corresponding to the maximum charge (nuclear charge) groups, corresponding to smaller values of the charge were always observed on the plates. The energies of these groups go as the square of the charge. This made it possible to determine with great accuracy the relative curves of the range-energy dependence for the N and O ions. Unfortunately, it was impossible to determine the equilibrium charge of ions for various energies in the course of the experiment since the ion energy spectrum before the stripping foil was unknown.

The foil was removed for the determination of range-energy dependence at low energies. As it had been shown in course of the experiment essentially singly and doubly charged ions arrived in that case at the plate, with the energy between 3 and 15 Mev. The ion beam was incident at a fixed angle of 30° to the emulsion surface. This made it possible to determine the actual track length measuring the horizontal projection only and excluded errors connected with the variation of the emulsion shrinkage coefficient.

The ion ranges were measured by means of a microscope of 2000× magnification with an accuracy of 0.3 μ. 200 — 300 tracks were measured for each energy value. The distribution of particle ranges, obtained for each value of energy, was represented in the form of a Gaussian curve from which the mean range and the half-width of the line were computed. The half-width at ion energies of ~ 100 Mev amounted to 3 Mev.



Range-energy dependence for N and O ions in Ilford E-1 emulsion. Solid curves and black circles represent the data of the present work; + — data of Chaminade et al.; dashes — curves calculated by Papineau; dots — curves calculated by Longchamp.

Ilford E-1 plates were used in the experiments. In experiments with low-energy ions, when tracks were a few micron long, NIKFI-Ia plates were used which, unlike the Ilford plates, are not coated with a protective gelatine layer. According to our measurements, carried out by means Po, ThC and ThC$'$ α-particles and N and O ions of various energies, the stopping power of the NIKFI-Ia emulsion is 10% less than that of Ilford E-1.

The obtained range-energy relations for N and O ions in the Ilford E-1 emulsion are shown in the figure. Data on the range of N and O ions with energy < 15 Mev obtained with the NIKFI-Ia emulsion are normalized to the ranges in the Ilford E-1 emulsion accounting for the difference in their stopping power. The accuracy of the determination of each point amounts to 5% for ions of more than 30 Mev and 10% for the ions of less than 15 Mev. The accuracy of the relative curve shape is, as it was mentioned above, considerably greater. It is limited only by range straggling and amounts to 2%.

The results obtained by us for the N ions in the range below 25 Mev are in a good agreement with the data of Reynolds and Zucker[6] and Chaminade et al.;[7] these are not shown in the figure for sake of clarity. For high energies the ranges obtained in Ref. 7 are slightly larger than the results of the present work.

Theoretical curves for N and O ions calculated by Longchamp[1] and Papineau[2] are also shown in the figure. For low energies the results of Papineau are in good agreement with the experimental data while for higher energies the curves calculated by Longchamp represent a better approximation. In the calculations of Papineau it was assumed that nitrogen ions lose all their electrons at ~ 230 Mev and the oxygen ions at ~ 320 Mev; Longchamp assumed that both N and O ions are completely stripped at 22 — 25 Mev already. According to our estimates, nitrogen ions reach their maximum charge at 70 — 90 Mev, oxygen ions at 100 — 120 Mev.

We are obliged to Academician I. V. Kurchatov for his advice given in the discussion of the present work. We are indebted to A. M. Bidnaia, K. I. Merkulova and L. E. Soboleva for the measurements of ion ranges in the photographic plates.

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THE FISSION OF URANIUM AND PROTACTINIUM AT HIGH EXCITATION ENERGIES

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The fission of uranium and protactinium at initial excitation energies of up to 100 Mev is considered. The analysis is based on comparison of the yields of the uranium isotopes $U^{236}-U^{232}$ and protactinium isotopes $Pa^{231}-Pa^{227}$ (equal values of the parameter Z^2/A) produced in the disintegration of uranium U^{238} by 340 Mev protons. The results of the analysis point to the emission nature of uranium and protactinium fission in the indicated range of initial excitation energy.

INTRODUCTION

A series of studies carried out in our laboratory[1-4] have investigated the mechanism of the fission of heavy nuclei at high initial energies of excitation. The subjects of these investigations have been three fissionable heavy nuclei of distinctly different compostion ($_{92}U$, $_{83}Bi$, $_{74}W$).

Analysis of the fission events in tungsten[1] showed that the cross-sections for pure emission fission for nuclei with charge $Z \leq 71-72$ become sufficiently small that fission from a level somewhat higher than the level of emission fission begins to compete successfully. In the work studying the fission of Bi (Ref. 2) with 660-Mev protons an evaluation was carried out of the threshhold for emission fission for a series of elements (from $Z = 72$ to $Z = 82$). The values obtained for the threshholds for emission fission agree well with the corresponding values calculated by Gol' danskii.[5] Finally, an analysis of fission events in uranium ($E_p = 660$ Mev) showed that the fission process at high initial energies of excitation can be divided into two stages. In the first stage the very highly excited uranium nucleus cools off by the evaporation of many particles (primarily neutrons). Fission occurs as the final act of the cooling process and takes place after the original nucleus has lost practically all of its initial excitation.

Thus heavy nuclei (U, Bi, W), having high excitation energies U_0, emit nucleons in the first stage of cooling down. Because of the Coulomb barrier, the most probable process is the evaporation of neutrons. Owing to the evaporation of neutrons, the critical value of the barrier for fission U_f gradually decreases while the binding energy ϵ_n of a neutron in the residual nucleus increases. As a result of this process there comes a time when the critical value for fission becomes equal to the binding energy of a neutron in the residual nucleus ($U_f = \epsilon_n$).

According to the estimates made by Gol'danskii,[5] in order to get a condition $U_f = \epsilon_n$, tungsten must emit 16 neutrons, bismuth must emit 6, while for uranium, in general, no preliminary evaporation of neutrons is necessary. The initial energies of excitation necessary for the evaporation of the required number of neutrons with subsequent fission of the elements in question are as follows:

$$U_0(\mathrm{W}) = 210\,\mathrm{Mev};\ U_0(\mathrm{Bi}) = 61\,\mathrm{Mev};\ U_0(\mathrm{U}) = 5.2\ \mathrm{Mev}.$$

Since a cross-section for emissive fission can be regarded as the cross-section for the emission before fission of the required number of neutrons ($N \geq N_{fis} = A_0 - A_{fis}$), the number of which is determined by the relation $U_f \leq \epsilon_n$, and the relative probability for the emission of neutron to that for the emission of a proton decreases rapidly both with the increase in initial energy excitation and with decreasing charge of the fissioning nucleus, it is obvious that the cross-section for emissive fission should decrease strongly with decreasing charge of the fissioning nucleus. Nuclei lighter than tungsten must lose more than 16 neutrons to achieve the conditions for emissive fission (in the case that a charged particle is not emitted), and such a process is extremely unlikely at an initial energy of excitation $U_0 > 210$ Mev. Because of these reasons the cross-section for emissive fission becomes so small that fission from a level somewhat higher than the level for emissive fission begins to compete.[1] For nuclei in the region of bismuth the conditions for emissive fission $U_f \leq \epsilon_n$ are reached very easily, since at an initial excitation energy

$U_0 \sim 60$ Mev the most probable process is the emission of neutrons only. Thus the overwhelming majority of fissions are emissive.[2]

The most interesting situation in our opinion lies in the fission of highly excited nuclei ($U_0 \gg U_f$) with charge $Z > 88$. For this group of heavy nuclei, in agreement with the considerations of Gol'danskii, it is not necessary to emit neutrons in order to reach the condition $U_f = \epsilon_n$. For this reason the usual picture of emissive fission for these nuclei loses its meaning.

At an initial energy of excitation $U_0 \gg U_f$ there are possible three different variations of fission: (A) fission from the top level of excitation $E(E = U_0)$, (B) fission from an intermediate stay of excitation ($U_f < E < U_0$), (C) fission from a low state of excitation ($U_f < E \approx \epsilon_n$).

Fission from the low state of excitation will be designated "deep" emissive fission to emphasize that fission is the final act of cooling and occurs after the initial nucleus has gotten rid of almost all the initial energy of excitation by the preliminary evaporation of particles. Analysis of the results of the above mentioned work[5] led us to the conclusion that the fission of uranium, when it is initially very highly excited, proceeds according to this "deep" emissive fission mechanism. Additional confirmation of our conclusions about this "deep" emissive fission can be obtained, it seems to us, by analysis of the experimental data of Lindner and Osborne.[6] We now proceed to examine this work.

ANALYSIS OF THE RESULTS OF LINDNER AND OSBORNE ON THE YIELDS OF SPALLATION PRODUCTS IN THE 340 MEV PROTON BOMBARDMENT OF URANIUM

The authors of this work,[6] studying the spallation products from the bombardment of uranium with 340 Mev protons, were able to determine the yields (among others) of isotopes of uranium and protactinium to mass $A = 237$. The cross-sections for the formation of the different isotopes of uranium and protactinium are given in Table I.

Since the masses of the lightest isotopes (U^{228}, Pa^{227}) differ from the original nucleus U^{238} only by 10 mass numbers it is reasonable to assume that the original energy of excitation U_0 for those nuclei which, as a result of the cascade-evaporation process, lead to the formation of the given isotopes, does not exceed 80 — 100 Mev. The evaporation of a proton at such initial energies of excitation can be neglected. It follows, therefore, that those cases in which isotopes of protactinium are formed as a result of subsequent evaporation involve a first-stage cascade process in which two protons are ejected; in nuclei which gave rise to isotopes of uranium, the cascade process caused one proton to be knocked out.

An additional argument in favor of this picture is obtained from a comparison of the yields of two isobars (uranium and protactinium). Since the yields of the protactinium isobars ($A < 236$) are greater than the yields of the corresponding uranium isobars (Table I) it is very unlikely that one of the two protons arose from an evaporation process (at an initial excitation energy $U_0 \approx 80 - 100$ Mev the evaporation of only neutrons is a more probable process than the evaporation of one proton and the rest neutrons). Thus, if the original fast proton collides with a neutron and then (after several collisions) both leave the nucleus, the result is an excited U^{237} nucleus, which, by subsequent cooling, gives some isotope of uranium (depending on the original energy of excitation). If, however, the original proton collides with a proton and both (likewise after several collisions) leave the nucleus, then the result is an excited Pa^{237} nucleus which, by subsequent evaporation, gives various isotopes of protactinium. The relative probability of these two processes can be determined from the following equation:

TABLE I

A	Cross-Section for the Formation of Uranium Isotopes, millibarns	Cross-Section for the Formation of Protractinium Isotopes, millibarns	$\sigma(Pa)/\sigma(U)$
237	50	35	0,7
236	30	25	0.84
235	20	21±2	1,05
234	10	14	1.4
233	6	12	2
232	3	8,7±1	2.9
231	1,1	7	6.4
230	0,35 ±0,12	5,1±0,5	14.5
229	0,06 ±0.005	2,5	42
228	0.038±0.002	1,7±0,2	45
227		0.71±0.06	
	$\Sigma\sigma = 121$ mbn.	$\Sigma\sigma = 132$ mbn.	

$$\frac{N\,(U_{92}^{237})}{N\,(Pa_{91}^{237})} \approx \frac{A - Z}{Z} \left(\frac{\sigma_{pn}}{\sigma_{pp}}\right)_{\text{ins. nucl.}} \tag{1}$$

In order to evaluate the ratio $(\sigma_{pn}/\sigma_{pp})_{\text{ins. nucl.}}$ we start from the known relation of Goldberger,[7] according to whose calculations the correction to the cross-section for isotropic scattering is given by the relation

$$\sigma_{\text{ins. nucl.}} = \sigma_{\text{free}}\left(1 - \frac{7}{5}\frac{P_F^2}{P_i^2}\right) = \sigma_{\text{free}}\left(1 - \frac{7}{5}\frac{E_F}{E_i}\right). \tag{2}$$

Here E_F is the nucleon energy corresponding of the Fermi momentum, E_i is the energy of the incoming nucleus inside the nucleus ($E_i = E_0 + v$), where v is the depth of the potential hole. The values of the Fermi energies E_F for a neutron and proton Fermi gas in a uranium nucleus are equal to $E_F = 27$ Mev (neutron gas) and $E_F = 21$ Mev (proton gas).

The change in the cross-section σ_{pp} on going from free to bound particles can be determined directly from Eq. (2). Since the cross-section $\sigma_{pp} = \sigma_{nn}$ we used for the calculations the cross-section σ_{nn} for neutrons having an average effective energy $\overline{E}_{n_0} = 380$ Mev,[8]

$$\sigma_{nn} = (20 \pm 1.4)\cdot 10^{-27}\ \text{cm}^2; \qquad (\sigma_{pp})_{\text{ins. nucl.}} = (\sigma_{nn})_{\text{free}}\left(1 - \frac{7}{5}\frac{E_F}{E_i}\right) = (18.4 \pm 0{,}7)\cdot 10^{-27}\ \text{cm}^2. \tag{3}$$

To evaluate the cross-section $(\sigma_{pn})_{\text{ins. nucl.}}$ we proceed from the following consideration: (A) the differential n-p scattering cross-section at $E_{n_0} = 380$ Mev is practically independent of the angle[9] over over a large angular interval (50 — 130°); (B) there is a significant increase in the differential n-p scattering cross-section in the angular regions 0 — 50° and 130 — 180°, i.e., in the region where the imparted momentum is small and the Pauli exclusion principle is most important; (C) we can separate out from the total n-p scattering cross-section the isotropic part for which the Goldberger relation is valid

$$(\sigma_{np})_{\text{isotr.}} = \sigma_{np}(\Phi = 90^\circ)\cdot 4\pi = 29\cdot 10^{-27}\ \text{cm}^2. \tag{4}$$

Thus the lower limit for the n-p scattering cross-section inside the uranium nucleus can be obtained from the relation

$$(\sigma_{np})_{\text{ins. nucl.}} \geqslant (\sigma_{np})_{\text{isotr.}}\left(1 - \frac{7}{5}\frac{E_F}{E_i}\right) = 26\cdot 10^{-27}\ \text{cm}^2. \tag{5}$$

An upper limit to this scattering cross-section inside the uranium nucleus can be obtained from the assumption that the total n-p scattering cross-section for free particles is isotropic

$$(\sigma_{np})_{\text{ins. nucl.}} < (\sigma_{np})_{\text{free}}\left(1 - \frac{7}{5}\frac{E_F}{E_i}\right) = 36\cdot 10^{-27}\ \text{cm}^2. \tag{6}$$

The cross-section for n-p scattering by free particles at an effective neutron energy $\overline{E}_{n_0} = 380$ Mev (Ref. 9) is $\sigma_{np} = (40 \pm 4)\times 10^{-27}\ \text{cm}^2$. The actual value of σ_{np} inside the nucleus must lie within the indicated limits.[4,6]

$$36\cdot 10^{-27}\ \text{cm}^2 > (\sigma_{np})_{\text{ins. nucl.}} \geqslant 26\cdot 10^{-27}\ \text{cm}^2. \tag{7}$$

Using Eqs. (3) and (7) we can determine the numerical value of the relation (1)

$$\frac{N(\text{U}^{237})}{N(\text{Pa}^{237})} \approx \frac{146}{92}\,\frac{(31 \pm 5)\cdot 10^{-27}}{(18.4 \pm 0.7)\cdot 10^{-27}} = 2.7 \pm 0.5.$$

Thus the total number of excited uranium nuclei (which during subsequent cooling either undergo fission or emit neutrons giving the observed isotopes of $\text{U}^{237} - \text{U}^{238}$) should be about 2.7 times as great as the total number of excited protactinium nuclei (which also either undergo fission or else split to give isotopes of protactinium $\text{Pa}^{237} - \text{Pa}^{227}$). Since the integrated yield of the uranium isotopes $\sum_{A=228}^{237} \sigma(\text{U}_{92}^{A})$ is approximately equal to the integrated yield of protactinium isotopes $\sum_{A=227}^{237} \sigma(\text{Pa}_{91}^{A})$, it follows that the integrated probability of spallation of the protactinium nuclei is 2.7 times greater than the integrated spallation probability of uranium. In the process of cooling of both uranium and protactinium there is primarily a competition between two processes: emission of neutrons and fission. Thus it is clear that the smaller integral probability of spallation of U^{237} indicates a large integral probability of fission of U^{237} nuclei in comparison with Pa^{237} nuclei.

However, the mere fact of the higher integrated probability of fission of U^{237} nuclei in comparison with Pa^{237} nuclei does not allow one to establish by which of the possible schemes the fission of these nuclei occurs, since both in a case of over the barrier fission [(A) and (B), page 269] and in the case of "deep" emission fission the probability of U^{237} fission should be greater than the probability of Pa^{237} fission. In order to establish by which of the possible paths the fission procedes it is necessary, in our opinion, to introduce into the discussion the fission parameter Z^2/A. Table II presents the values of Z^2/A for isobars of uranium and protactinium.

The values of the parameter Z^2/A for corresponding isobars of uranium and protactinium differ by about 2%. However, if the isotopes of uranium are compared with isotopes of protactinium of 5 mass numbers lower (following the arrows in Table II), it is seen that the values of the Z^2/A parameter for such pairs are the same.

TABLE II

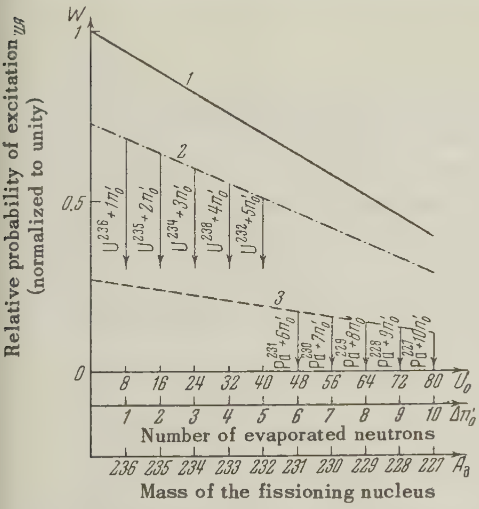
A	$Z^2/A\ (U_{92}^{A})$	$Z^2/A\ (Pa_{91}^{A})$
237	35.7 ←	34.9
236	35.86	35.06
235	36.02	35.22
234	36.18	35.38
233	36.34	35.54
232	36.5 ←	→ 35.7
231	36.66	35.86
230	36.82	36.02
229	36.98	36.18
228	37.14	36.34
227	37.3	→ 36.5

Using the numbers presented in Table I we can determine the relative yields from spallation for these indicated pairs. Table III shows that these ratios are practically constant for all the indicated pairs.

Further analysis of the data[6] must take into account the fact that, as a result of the cascade, there results a whole spectrum of excitation energies. It is known that the yield of excited nuclei decreases as the initial energy of excitation increases. According to indication given by the authors of the work being examined,[6] the spectrum of initial energies of excitation in the region 0 — 80 Mev changes in such a way that the yield of nuclei excited to energies $U_0 \sim 80$ Mev falls by a factor of 2 — 3.

The figure shows a possible spectrum of initial energy of excitation on the assumption that the yield of excited nuclei in the given interval of energy varies linearly. The solid line gives the total yield of uranium and protactinium nuclei, the dot-dash line gives the yield of uranium nuclei and the dashed line gives the yield of protactinium nuclei. The relative yield of excited uranium and protactinium nuclei is determined from relation (1).

Since we assume that as a result of the cascade process there is formed either a U^{237} or Pa^{237} nucleus, and that the final nuclei after spallation (for equal values Z^2/A) are shifted by 5 mass units, it follows that Pa^{237} nuclei which give as a result of neutron evaporation an isotope of protactinium (for one of our comparable pairs) have initial energies of excitation 40 Mev higher than the initial energy of excitation of U^{237} nuclei which give the corresponding isotope of uranium (it is assumed that each neutron emitted carries off an energy $\Delta U_0 = \epsilon_n + 2\overline{T} \approx 8$ Mev.



Distribution of uranium nuclei, U^{237}, and protactinium nuclei, Pa^{237}, as a function of initial excitation energy U_0. Δn_0 is determined from the relation $\Delta n_0 = U_0/(\epsilon_n + 2\overline{T}) \approx U_0/8$; $A_b = 237 - (U_0/8)$. 1 — $N(U^{237} + Pa^{237})$; 2 — $N(U^{237})$; 3 — $N(Pa^{237})$.

Analysis of the numbers presented in the diagram shows that the number of U^{237} nuclei having initial energies of excitation in the interval 8 — 40 Mev and giving as a result of neutron evaporation the isotopes of $U^{236} - U^{232}$ is 4.2 times more abundant than Pa^{237} nuclei having initial energies of excitation in the interval 48 — 80 Mev which lead as a result of neutron evaporation (6 — 10) to the protactinium isotopes $Pa^{231} - Pa^{227}$.

If the fission of U^{237} and Pa^{237} occurs according to the process of "deep" emission fission, i.e., if fission occurs when the relation $U_f < E \approx \epsilon_n$ is satisfied, then the probability for fission should be determined by the final value of the parameter Z^2/A. Thus, in the case of "deep" emission fission, the following relations should be satisfied for our pairs of nuclei with equal Z^2/A (aside from the pair $U^{237} - Pa^{232}$, in which the nucleus U^{237} is formed directly in the cascade process and has, by the assumption made above, zero excitation energy):

$$\frac{W(U^{237})_{U_0}}{W(Pa^{237})_{U_0+40}} \approx 1; \quad \frac{|N(Pa^{237})|_{U_0+40}}{|N(U^{237})|_{U_0}} \quad \frac{\sigma_{\text{spall}}(U_{92}^{A})}{\sigma_{\text{spall}}(Pa_{91}^{A-5})} \approx 1; \tag{8}$$

In this equation W is the probability for spallation, $N(Pa^{237})_{U_0+40}$ is the number of Pa^{237} nuclei having initial excitation energy $U_0 + 40$ Mev, $N(U^{237})_{U_0}$ is the number of U^{237} nuclei having initial excitation energy U_0, $\sigma_{\text{spall}}(U_{92}^{A})$ is the cross-section for the formation of uranium spallation products with the given value of A, $\sigma_{\text{spall}}(Pa_{91}^{A-5})$ is the cross-section for the formation of protactinium spallation products having mass numbers 5 lower than the corresponding uranium spallation products. The results of the test of the relation (8) are presented in Table IV.

TABLE III

Yield of Isotopes of U, millibarns		Yield of Isotopes of Pa, millibarns		Relative Yields
A	σ	A	σ	$\sigma(U^A)/\sigma(Pa^A)$
237	50	232	8 ± 1	5.7
236	30	231	7	4.3
235	20	230	5.1 ± 0.5	3.94
234	10	229	2.5	4
233	6	228	1.7 ± 0.2	3.54
232	3	227	0.71 ± 0.06	4.2

The fourth column of the table gives the ratio of the number of excited Pa^{237} nuclei giving isotopes of protactinium (for the pairs being considered) to the number of excited U^{237} nuclei giving the corresponding isotopes of uranium (the numbers are taken from the curves in the figure). The fifth column gives the ratio of the yields of spallation products for the pairs being considered (the values are taken from Table III.).

We shall now try to analyze relation (8) from the point of view of "over the barrier fission," i.e., assuming that fission competes with neutron emission during the whole process of cooling. This cooling process can be broken up into steps of size $\Delta U_0 = \epsilon_n + 2\overline{T} = 8$ Mev. At each of these steps the nucleus can either undergo fission or emit a neutron. The relationship (8) for the first of our pairs ($U^{236} - Pa^{231}$) gives a value of unity (Table IV). From the point of view of "over the barrier fission" this means that the relative probability of emitting a neutron from an excited uranium nucleus is considerably smaller than the relative probability of emitting a neutron from an excited protaction nucleus. We shall elaborate on this in more detail.

TABLE IV

Investigated Pairs	$U_0(U^{237})$	$U_0(Pa^{237})$	$\dfrac{N(Pa^{237})_{U_0+40}}{N(U^{237})_{U_0}}$	$\dfrac{\sigma_{spall}(U^A_{92})}{\sigma_{spall}(Pa^{A-5}_{91})}$	$\dfrac{N(Pa)_{U_0+40}\,\sigma(U^A_{92})}{N(U)_{U_0}\,\sigma(Pa^{A-5}_{91})}$
U^{236}— Pa^{231}	8	48	0.257	4.3	1.08
U^{235}— Pa^{230}	16	56	0.253	3.94	0.98
U^{234}— Pa^{229}	24	64	0.234	4	0.94
U^{233}— Pa^{228}	32	72	0.236	3.54	0.85
U^{232}— Pa^{227}	40	80	0.216	4.2	0.9

The number of U^{236} nuclei which have formed as a result of the emission of one neutron from U^{237} nucleus excited originally by an amount $U_0 = 8$ Mev is equal to

$$N(U^{236}_{92}) \approx \omega N(U^{237}_{92})_{U_0=8\ \mathrm{MeV}}, \tag{9}$$

where ω is the relative probability of emission of a neutron from a uranium nucleus. The cooling process of Pa^{237}_{91} having an initial exciting energy $U_0 = 47$ Mev proceeds in 6 steps ($n_{step} = U_0/(\epsilon_n + 2\overline{T}) \approx 6$), in each of these steps there being the probability of an emission of a neutron. The number of Pa^{231}_{91} nuclei is determined by the relation

$$N(Pa^{231}_{91}) \approx \omega_1^6 N(Pa^{237}_{91})_{U_0=48}. \tag{10}$$

The relative yields of Pa^{231} and U^{236} are determined by the relations

$$\frac{W(Pa^{231}_{91})}{W(U^{236}_{92})} = \frac{\omega_1^6 N(Pa^{237}_{91})_{U_0=48}}{\omega N(U^{237}_{92})_{U_0=8}} = \frac{\sigma_{spall}(Pa^{231}_{91})}{\sigma_{spall}(U^{236}_{92})}, \tag{11}$$

$$\frac{\sigma_{spall}(U^{236}_{92})\cdot N(Pa^{237}_{91})_{U_0=48}}{\sigma_{spall}(Pa^{231}_{91})\, N(U^{237}_{92})_{U_0=8}} = \frac{\omega}{\omega_1^6}. \tag{11'}$$

TABLE V

Investigated Pairs	Actual Value of the Left Side of (11')	Value of the Left Side of Eq. (11') on the Assumption of "Over the Barrier Fission"
U^{236}—Pa^{231}	1.08	1
U^{235}—Pa^{230}	0.98	0.445
U^{234}—Pa^{229}	0.94	0.197
U^{233}—Pa^{228}	0.85	0.087
U^{232}—Pa^{227}	0.9	0.039

The left side of (11') is nothing more than Eq. (8) written for the first of our pairs ($U^{236}_{92} - Pa^{231}_{91}$) and is equal to unity as is seen from Table IV. It thus follows that for "over the barrier fission" $\omega/\omega_1^6 = 1$. We can get numerical values for ω and ω_1 in the following way. We have

$$\frac{\sigma_{spall}(U^{236}_{92})}{\sigma_{spall}(Pa^{236}_{91})} = \frac{N_{spall}(U^{236}_{92})}{N_{spall}(Pa^{236}_{91})} = \frac{\omega N(U^{237}_{92})_{U_0=8}}{\omega_1 N(Pa^{237}_{91})_{U_0=8}}. \tag{12}$$

From relation (1) we have $N(U^{237}_{92})_{U_0=8}/N(Pa^{237}_{91})_{U_0=8} \approx 2.7$, and from Table I $\dfrac{\sigma_{spall}(U^{236}_{92})}{\sigma_{spall}(Pu^{236}_{91})} \approx 1.2$ and thus $\omega/\omega_1 = 1.2/2.7 = 0.445$. Thus $\omega = 0.38$ and $\omega_1 = 0.85$. Knowing the values of ω and ω_1 we can calculate the left part of (11') for all 5 pairs that interest us on the assumption of "over the barrier fission;"

$$\frac{|N(\mathrm{Pa}^{237})|_{U_0+40}}{|N(\mathrm{U}^{237})|_{U_0}} \cdot \frac{\sigma_{\mathrm{spall}}(\mathrm{U}_{92}^{A})}{\sigma_{\mathrm{spall}}(\mathrm{Pa}_{91}^{A-5})} = \frac{\omega^{\alpha}}{\omega_1^{\beta}}; \qquad (13)$$

$$\alpha = U_0/(\varepsilon_n + 2\overline{T}) \approx U_0/8; \quad \beta = (U_0 + 40)/(\varepsilon_n + 2\overline{T}) \approx (U_0 + 40)/8. \qquad (14)$$

The results of these calculations are presented in Table V.

The data of Table V show that the assumption of "over the barrier fission" gives results (column 3) which, in our opinion, cannot be reconciled with the experimental data (column 2). On the other hand, the results of the analysis presented in Table IV, show that relation (8) is satisfied sufficiently well and therefore this can be taken as an additional argument for "deep" emission fission of uranium and protactinium nuclei in the initial energy of excitation range under consideration.

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SOVIET PHYSICS JETP VOLUME 6(33) NUMBER 2 FEBRUARY, 1958

ON THE CROSS SECTION FOR PRODUCTION OF MULTIPLY CHARGED PARTICLES IN THE INTERACTION BETWEEN PROTONS AND NUCLEI

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Nuclear emulsions were used to study the bombarding proton energy and atomic weight dependence of the cross section for production of multiply charged ($Z \geq 4$) particles in nuclear disintegrations due to fast protons. The cross section for production of particles possessing an energy greater than 1 – 2 Mev per nucleon varies from 3 to 12×10^{-27} cm^2 for incident particle energies lying between 300 – 660 Mev if heavy nuclei (Ag, Br) are disintegrated and remains equal to 2×10^{-27} cm^2 if light nuclei (C, N, O) are disintegrated. For a given energy of the incident particles the cross section of the reaction increases with increase of the mass number of the target nuclei.

I. INTRODUCTION

IT is presently well known that the cross section for the reaction leading to the production of multiply charged particles increases with the increase of energy of the bombarding particles (in the range $10^2 - 10^3$ Mev). Nevertheless, neither the functioning of this dependence nor the absolute value of the cross section for the production of multiply charged fragments is yet known. When high energy particles produce disintegrations emitting multiply charged fragments, the dependence of the cross section for this reaction on the atomic number of the target nuclei, established by radiochemical methods,[1] is the same for all fragments: the cross section for such fragment production varies inversely with the atomic number of the target nuclei. Nevertheless, as has been noted previously,[2] a significant portion of the cross section for the fragment-production reaction can be accounted for the stable isotopes of light elements which are not detected by radiochemical methods, and it is not clear a priori whether the indicated dependence is valid for the total cross section of the reaction.

This article presents experimental data concerning the dependence of the fragment-production cross section (σ_{frag}) on the incident proton energy and on the mass number of the target nuclei, obtained with the aid of nuclear emulsions. The fine-grained nuclear emulsions used (Type P-9 emulsions prepared in the laboratory of N. A. Perfilov) provided the means for reliable identification of multiply charged particles for $Z \geq 4$.

II. DEPENDENCE OF THE CROSS SECTION ON THE ENERGY OF THE INCIDENT PROTONS

The cross sections for the production of multiply charged particles in the interactions between protons of 300 — 660 Mev and the light and heavy nuclei in the emulsions were determined by two methods. In the first method, a plate was irradiated with a known perpendicular proton current and the total number of disintegrations producing fragments that occurred within a given area was counted. In this case, $\sigma_{frag} = N/N_n N_p$ (where N is the number of disintegrations producing fragments per square centimeter; N_n is the number of nuclei of Ag, Br, or C, N, O, per square centimerer; N_p is the proton current per square centimeter).

In the second method, an emulsion was irradiated with protons of known energy, and the percentage of inelastic interactions with Ag and Br nuclei in the emulsion which led to fragments (q) was determined. The desired cross section was found by multiplying the known cross section for interaction of protons with Ag and Br by the factor q. Nuclear disintegrations with fragments, as observed in emulsions, can be explained as disintegrations of either heavy nuclei (Ag, Br) or of light nuclei (C, N, O). Differentiation between these two groups of disintegrations was accomplished by using two criteria: total charge of the particles in the disintegration and the presence or absence of tracks of residual nuclei. The presence of residual nucleus tracks is characteristic of disintegrations of heavy nuclei with multiply charged particles

The results obtained with nuclei of Ag, Br, and C, N, O are shown in the figure below. It is necessary to note that the cross section data relate only to fragments with $Z \geq 4$ having paths longer than 12 — 15 microns. This latter condition is set by the limit of possible identification — i.e., to be identifiable, fragments must have energies greater than 1 — 2 Mev. Quite noteworthy is the difference between the energy dependence of σ_{frag} in the case of heavy nuclei and in the case of light nuclei. While the fragment production cross section increases sharply with the energy in the case of Ag and Br, it is almost invariant with respect to energy in the case of C, N, O. This circumstance can be explained by the mechanisms of fragment production that take place for light nuclei and for heavy nuclei.

It is interesting to note that at the same time the cross section for the reaction producing splitting of Ag or Br nuclei into two fragments of nearly equal mass (such events, which resemble the fission of heavier nuclei, are occasionally observed in emulsions) is apparently not strongly dependent on the energy of the bombarding protons. The cross section for such fission of Ag and Br, determined by observations on the same plates, has a value $\sim 0.4 \times 10^{-27}$ cm^2 for incident particle energies of 300 and 660 Mev.

III. DEPENDENCE OF THE CROSS SECTION ON THE MASS NUMBER OF THE TARGET NUCLEUS

The dependence of cross section on target mass number was studied using direct contact between the target and the emulsion. Plates of Al, Cd and Pb about one millimeter thick were placed on the nuclear emulsion and irradiated with a proton beam at an angle of 12 degrees with respect to the emulsion surface The tracks made by particles with $Z \geq 4$, which entered the emulsion through its surface, were then counted.

The fragments in question were produced by disintegration of the elements under study. Since the targets used are infinitely thick to these fragments, and since the emission of fragments can be considere equally probable with respect to the dividing plane between emulsion and metal plate, the appropriate radiographic formulas are applicable.

Target	Fragments Observed per cm² of Emulsion	$n\sqrt{A}$
Al	50±5	260
Cd	50±5	527
Pb	63±6	905

As the result of formulas given in the work of Curie and Faraggi,[3] it is possible to write the following expression for the effective cross section of the interaction leading to the emission of multiply charged fragments:

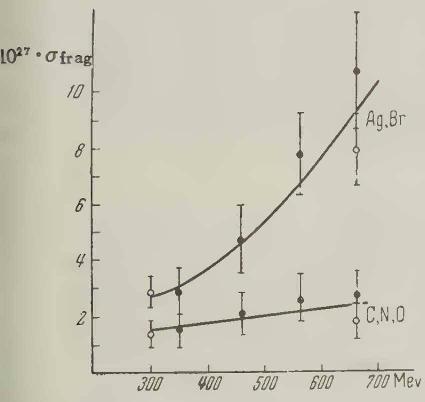
$$\sigma_{\mathrm{frag}} = 2.08 \cdot 10^{-20}\,(A/N_p)\,n/\psi\,(\bar{R} - \rho),$$

where A is the atomic weight of the target material, N_p is the proton flux, n is the observed number of fragments per square centimeter of emulsion, ψ is the magnitude of the target material penetrability, which is a measure of its stopping power, $\bar{R}$ is the average particle range, and ρ is the minimum range for registration. If it is assumed that the fragment range spectra are identical for Al, Cd and Pb, and that the target material penetrability is the same for these fragments as for α-particles,[3] i.e., $\psi = \sqrt{A}$, then:

$$\sigma_{\mathrm{frag}} \sim n\sqrt{A}.$$

The table lists the experimental values of n, obtained with 660 Mev protons, and the computed values of $n\sqrt{A}$. If the indicated assumptions are correct, the resultant dependence of $n\sqrt{A}$ on A can serve as direct evidence of the increase of the cross section for fragment production with increasing target-material mass number. Qualitative arguments in support of the assumptions made are given below.

Owing to the anisotropic angular distribution of the studied fragments in the laboratory system, the assumption of equal probability for fragment emission with respect to the dividing plane between emulsion and target material is known to be incorrect. However, in the case under consideration, it is necessary only to be sure that the uncertainty thus introduced does not change the character of the dependence of $n\sqrt{A}$ on A. Investigation of the angular distribution of fragments from disintegration of various nuclei has shown that the degree of anisotropy of the angular distribution varies inversely with the atomic weight of the target nucleus. This circumstance leads to a relative increase in the magnitude of $n\sqrt{A}$ in the case of an aluminum target, and therefore cannot alter the basic conclusion concerning the increase of σ_{frag} with increasing A.



Dependence of σ_{frag} on the energy of the bombarding protons. Open circles — cross sections found by the direct method ($\sigma_{\mathrm{frag}} = N/N_n N_p$); black circles — cross sections found by the indirect method ($\sigma_{\mathrm{frag}} = q\sigma_{\mathrm{AgBr}}$).

The principal uncertainty in the results obtained arises from the lack of knowledge concerning the range distribution of the fragments under study. The assumption made above concerning the identical nature of the fragment range spectra for disintegrations of Al, Cd and Pb nuclei is based on the similarity found in the range distributions for fragments from Al, Cd and Pb, as recorded in the emulsion. If one also takes into account the fact that the average observed range for fragments of $Z \geq 4$ is greater in the emulsion for those coming from C, N, O nuclei than for those coming from Ag and Br nuclei, in the emulsion, then one can conclude that the increase of $n\sqrt{A}$ with increase of A is in fact due to the increase of σ_{frag}. This is confirmed by direct measurements of σ_{frag} for C, N, O and Ag, Br in the emulsion (see figure).

The most interesting inference that can be drawn from the results given above is that with increasing target mass number there is an increase in the fraction of stable isotopes relative to all the emitted fragments, since it is known from radiochemical work[1] that the yield of radioactive isotope fragments decreases with increasing target mass number, while the overall cross section for the reaction increases, as was determined in the work in question. However, one must bear in mind that these conclusions relate to fragments having energies greater than 2 — 3 Mev per nucleon, so that correlation with radiochemical results cannot be completely valid, especially in the case of light nuclei.

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CLOUD CHAMBER INVESTIGATION OF THE ELECTRON-PHOTON COMPONENT OF EXTENSIVE AIR SHOWERS AT SEA LEVEL

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The energy spectra of the electronic component of extensive air showers containing various numbers of particles were measured at different distances from the axis. No dependence of the energy spectrum on the number of particles in the shower was detected. The energy spectrum becomes softer with increasing distance from shower axis. A significant discrepancy was found to exist between the fraction of high-energy electrons observed experimentally and that computed from cascade theory for various distances from the axis. The lateral distribution of the energy carried by the electronic component can be approximated by the function r^{-n}, where $n = 2 \pm 0.5$ for distances between 2 and 10 meters.

IT was demonstrated[1] that, in the central region of extensive air showers, nuclear-active particles of more than 10^{11} ev can generate, by means of π-mesons, the electronic component near the observation level. This conclusion was reached as the result of a study of the lateral distribution of the electronic component in the central region of extensive showers. As a further study of the generation mechanism of electronic component we investigated its energy characteristics, which shoul depend more strongly on the production mechanism than the lateral distribution. The energy spectra of the electronic component at various distances from the shower axis were measured in course of the present work for showers containing different numbers of particles.

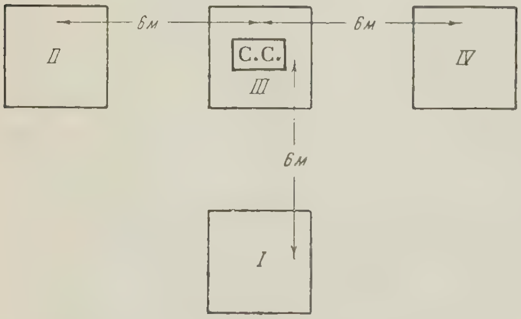

FIG. 1. Arrangement of hodoscope counter trays and the cloud chamber (C.C.)

The experimental arrangement consisted of a rectangular cloud chamber[2] with dimensions 60 × 60 × 30 cm and four groups of hodoscoped counters. A diagram of the positions of the hodoscope counter trays and the cloud chamber is shown in Fig. 1. The cloud chamber and the hodoscope were triggere by a system of counters discharged by the extensive air showers.* 2175 showers were recorded during 1189 hours of operation.

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* A detailed description of the triggering system and of the hodoscope arrangement will published in Ref. 3, where the effects connected with the innacuracy in the determination of the distance from axis will be discussed as well. A possible shift of the axis position towards smaller distances cannot evidently cause a substantial change in the results of the present work.

The position of axis r and the number of particles N in a shower were determined by the usual method[3,4] from the value of particle flux density ρ, recorded by three counter trays (II, III, and IV, Fig. 1) of the hodoscope. The distribution of all recorded showers with respect to intervals of r is shown in Fig. 2 (histogram).

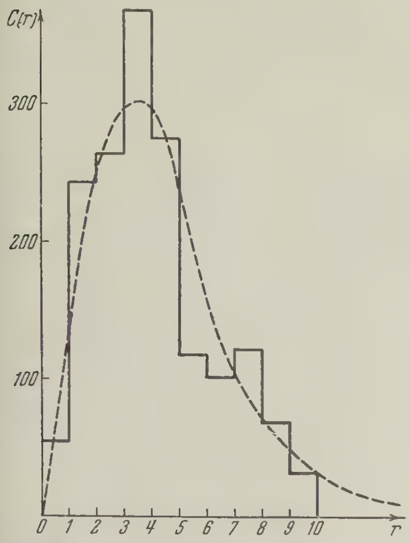

FIG. 2. Distance distribution of extensive air showers.

The energy of the electrons and photons was determined by means of a cloud chamber containing 7 lead plates, 0.5, 1.5, 2, 2, 2.5, 2, and 2.5 cm thick respectively, using cascade curves for lead calculated by Ivanenko[5] with allowance for scattering. Since the cascade curves[5] yield the total number of particles in a cascade shower and since the evaluation of shower photographs only the particles traveling forwards (within an angle < 180°) can be counted, we determined the fraction of the particles traveling forwards by using the angular distribution function of shower particles.[6] Data on the total number of particles which should have been observed between all plates of the cloud chamber, originating from a primary electron or photon of a given energy, are given in Table I.

The energy of electrons and photons in the energy range $10^8 - 10^9$ ev was determined from the total number of particles between the plates. This value, for a given primary energy, is subject to less fluctuations than any other characteristic of the cascade shower (number of particles at the maximum, position of the maximum) and represents therefore the best characteristic of the primary energy.

It was shown in Ref. 7 that the error of such a method of energy determination is close to statistical, and is therefore determined by the number of particles in the shower, i.e., it amounts to 20 — 30% for the investigated energy region. The above method of determination of the electronic shower energy was recently used in a number of investigations.[7-9] In the region of energies $\geq 2 \times 10^9$ ev the energy of electrons and photons was determined from the number of particles present at 11.5, 16.5, and 20.5 radiation units since it was difficult to count the number of particles at the maximum due to high density there. The accuracy of the energy determination in that region was also on the order of 30%.

TABLE I

Energy of the primary particle ($\times 10^8$ ev)	Electron	Photon	Energy of the primary particle ($\times 10^8$ ev)	Electron	Photon
1	3.2	2.7	9	24.9	21.5
2	6.0	5.1	10	27.3	24.0
3	8.9	7.4	20	51.3	48.0
4	11.6	9.5	40	100.0	96.0
5	14.7	12.0	60	150.0	147.0
6	17.8	14.0	80	200.0	198.0
7	20.3	16.5	100	250.0	250.0
8	22.6	19.0			

In order to avoid an uncertainty in the effective area of the chamber we considered only those electrons and photons that passed through the lid of the chamber. We also took into account the decrease in the effective area for inclined showers due to the fact that a part of the particles was outside the illuminated region of the chamber. The effective recording area varied between 0.12 and 0.10 m^2 for energy variation from 10^8 to 3×10^9 ev (for vertical showers the effective chamber area was equal to 0.15 m^2).

Three integral energy spectra of electrons and photons for three groups of extensive showers with $10^4 - 5 \times 10^4$, $5.1 \times 10^4 - 10^5$, and $1.1 \times 10^5 - 5 \times 10^5$ particles and for three intervals of distance from shower axis: $r \leq 3$, $3 < r \leq 6$, and $6 < r \leq 10$ m, were plotted to study the dependence of the energy spectrum of the electronic component on the number of particles and on the distance from axis. The results are showin in Fig. 3 (a, b, c).*

It can be seen from Fig. 3 that within the limits of statistical accuracy the energy spectrum at a given distance from the shower axis is independent of the number of particles in the shower. In Fig. 4, therefore, the spectra for the three distance intervals are averaged over all showers with number of particles between 10^4 and 5×10^5. It can be seen from the curves that the energy spectrum of electrons and photons depends on the distance from shower axis — the spectrum becomes softer with increasing distance.

From the obtained spectra we have determined what fraction of the total number of particles in extensive air showers comprises high-energy electrons and photons. We have found the following relation:

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*Average values, equal to $n+1$ (where $n < 10$), are shown in Figs. 3 and 4 for $\Pi_{\mathrm{exp}} (>E) = n$. The errors $\pm \Delta n$ are determined from the Poisson distribution.

$$R(r_1, r_2) = \int_{r_1}^{r_2} \rho_1(r)\, C(r)\, dr \Big/ \int_{r_1}^{r_2} \rho_0(r)\, C(r)\, dr,$$

where $\rho_1(r)$ is the density over 1 m^2 of electrons and photons with energy larger than 10^9 ev at the distance r from the axis, $\rho_0(r)$ is the density per square meter of all particles recorded by the counters at the same distance from axis, and C(r) is the number of extensive air showers recorded in the given distance interval (cf. Fig. 2). The value of this relation for showers with different N and for different distance intervals is given in the first three columns of Table II.

TABLE II

	Experimental results			Theoretical results	
N	10^4—$5\cdot10^4$	$5.1\cdot1\ {}^4$—10^5	$1.1\cdot10^5$—$5\cdot10^5$	$s=1$	$s=1.5$
$R(0.3)$	$(1.82\pm0.25)\%$	$(1.50\pm0.35)\%$	$(1.45\pm0.26)\%$	34%	12.5%
$R(3.6)$	—	$(0.17\pm0.13)\%$	$(0.45\pm0.11)\%$	5.3%	4.3%
$R(6.10)$	—	—	$(0.09\pm0.09)\%$	1.7%	2.0%

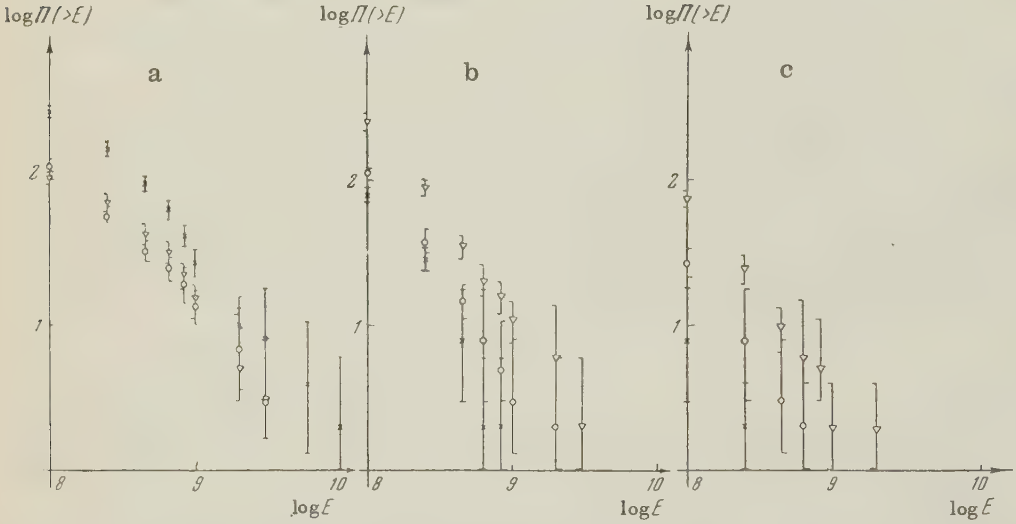

FIG. 3. Integral energy spectra of the electron-photon component for three groups of extensive air showers with the number of particles: × — $10^4 - 5 \times 10^4$, ⊙ — $5.1 \times 10^4 - 10^5$, ∇ — $1.1 \times 10^5 - 5 \times 10^5$, a — for r ≤ 3 m, b — for 3 < r ≤ 6 m, c — for 6 < r ≤ 10 m. Π (>E) — number of particles with energy >E, recorded by the cloud chamber in all showers, the axes of which fell in the corresponding distance interval.

The experimental data made it possible to estimate the energy flux carried by the electronic component of extensive air showers at various distances from axis. In calculating the energy flux we took into the account, besides the electrons and photons which underwent a multiplication in the lead plates of the cloud chamber, also the electrons and photons of comparatively low energy which were absorbed in the first or in the second plate. If the electron was absorbed in the first plate, which was 0.5 cm thick, its energy was assumed to be 10^7 ev; if the absorption of the particle took place in the second plate, 1.5 cm in thickness, its energy was estimated as 3×10^7 ev. These estimates are based on ionization losses only. Electrons and photons which produced pairs in the chamber were ascribed an energy of 5×10^7 ev. The contribution of all these particles to the energy flux is very small at distances less than 6 m from the shower axis and becomes noticeable only at distances > 6 m from the axis.

The lateral distribution of energy flux of the electronic component is represented in a log-log scale in Fig. 5 for three groups of showers with different number of particles.* It can be seen that within the limits of accuracy of the measurements the lateral distribution of energy flux is independent of the number of particles in the shower. If we approximate the dependence of the energy flux density on distance by the function r^{-n}, then n = 2 ± 0.5.

We compared the experimentally obtained energy spectra and fraction of high-energy particles with the corresponding values calculated from cascade theory in order to find how does the production mechanism of the electronic component influence its energy characteristics. It should be noted that, while the experimental high-energy spectra and fractions of high-energy particles were obtained by us for the electrons and photons together, calculations of the cascade theory refer to electrons only.

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*The effect of the mixing of the showers from different intervals due to the innacuracy of distance determination, described in Ref. 3, was taken into account in the calculations of the energy flux density at various distances from the axis.

We computed the energy spectra at various distances from the shower axis using the distance distribution of electrons with energy E calculated in Ref. 10. This distribution function was calculated in that reference for different values of the cascade parameter s and for distances down to zero.* Since our experimental spectra refer to distance intervals, we averaged the theoretical spectra over corresponding distance intervals using the experimental distance distribution of showers approximated by the solid curve in Fig. 2.

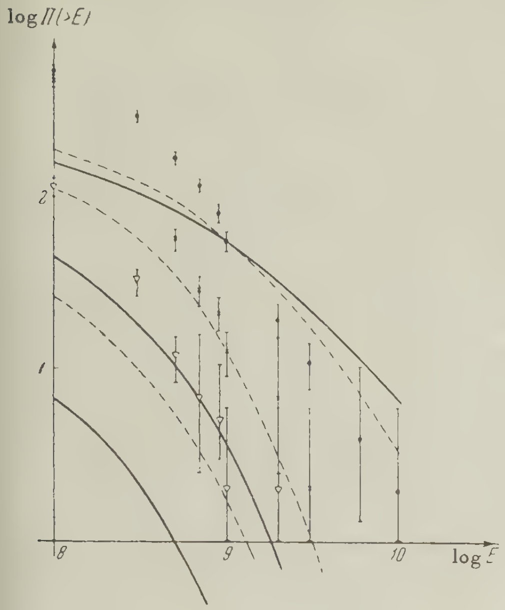

FIG. 4. Integral energy spectra of the electron-photon component of extensive air showers with the number of particles $10^4 - 5 \times 10^5$ for the distance intervals: ● — $r \leq 3$; × — $3 \leq r \leq 6$; ∇ — $6 < r \leq 10$ m; Electron spectra according to cascade theory are shown by the solid curve for $s = 1$, dotted — for $s = 1.5$. The experimental and the theoretical spectra are normalized for the energy 10^9 ev and $r \leq 3$ m.

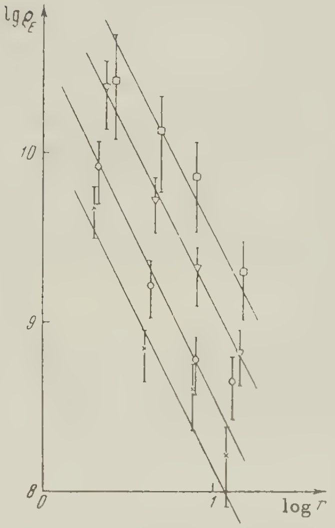

FIG. 5. Lateral distribution of the energy flux carried by the electronic component of extensive air showers. Energy flux density ρ_E is expressed in ev/m². Values of N: × — $10^4 - 5 \times 10^4$; ○ — $5.1 \times 10^4 - 10^5$; ∇ — $1.1 \times 10^5 - 5.1 \times 10^5$; □ — $5.1 \times 10^5 - 10^6$.

The integral energy spectra for $s = 1$ (solid curve) and $s = 1.5$ (dotted curve) are shown in Fig. 4. The experimental and the theoretical spectra are normalized for the energy 10^9 ev and $r \leq 3$. It can be seen that the experimental spectra are softer than the theoretical and are in better agreement with the value $s = 1.5$ than with $s = 1$. This fact is especially noticeable for small distances from the axis.

The fraction of high-energy electrons ($> 10^9$ ev) at various distances from shower axis was calculated by us in the following way: The total number of electrons in showers with $s = 1$ and $s = 1.5$ was determined from the cascade curves for air.[12] Their lateral distribution was calculated according to the approximation formula given by Greisen.[13] The number of electrons with energy $> 10^9$ ev in showers with $s = 1$ and $s = 1.5$ was determined from the well-known formula of cascade theory, without accounting for ionization losses. The lateral distribution of these electrons was obtained by means of calculations contained in Ref. 10. The fraction of high-energy electrons for showers with $s = 1$ and $s = 1.5$ averaged over distance intervals, are given in the two last columns of Table II.†

It can be seen from Table II that the value of the ratio $R(r_1, r_2)$ for showers recorded at sea level is considerably smaller than that computed from cascade theory. It is difficult to explain this fact on the basis of our present knowledge.

It is possible that in the central region of extensive air showers there is a large number of photons of comparatively low energy which are in equlibrium with high-energy electrons. These photons, undergoing a conversion in the dense covers placed above the counters (amounting, according to a crude estimate to ~ 1/3 t-units) and in the counter walls increase the flux density of low-energy particles. Nor can we exclude the possibility that the electronic component of extensive air showers recorded by us is generated by means of π-mesons possessing a very soft energy spectrum.

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*The distribution functions given in Ref. 10 for $s = 1$ agree within 10% with the corresponding functions of Ref. 11, and for $s = 1.5$ exceed them by 50% in the whole distance range.

†Analogous calculations of the fraction of high-energy electrons at various distances from the axis were carried out by Hazen[14] by means of cascade theory for the altitude of 3260 m. The ratio ρ_1/ρ_0 was calculated. The values obtained for $s = 1$ are in agreement with our data if one accounts for the difference between the values of the radiation unit for 3260 m and sea level.

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STATIONARY STATES OF ELECTRON-POSITRON SYSTEMS AND ANNIHILATION TRANSITIONS

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A scheme is set up which makes it possible to take into account the influence of an external field and the mutual interaction of electrons and positrons in the interaction of the electron-positron field with the radiation field. The theory is nonrelativistic with respect to the particle momenta in cases in which the photon energy is comparable with the rest energy of the electron. In cases in which the photon energy is much smaller than the rest energy of the electron, some relativistic corrections can be included in the theory by going to higher approximations. This scheme is applied to the stationary states of an electron-positron system and to the two-photon annihilation of a positron in a many-electron system. The equations of the Fock self-consistent field are obtained for an electron-positron system, and an expression is found for the probability of annihilation of a positron in a many-electron system. The special case of annihilation in helium is examined.

THE various processes of interaction of the electron-positron field with the radiation field are ordinarily treated in the approximation of the single-electron problem. A scheme of calculation in the configuration representation, designed for the treatment of many-body problems, has been set up by Shirokov.[1] This scheme was applied by Tumanov and Shirokov to derive the equation of positronium[2] and by Tumanov to the study of the two-photon annihilation of positronium.[3] The only case treated in these papers, however, is that of no external field.

In order to set up a scheme permitting us to include the effect of the external field and of the mutual interaction of the electrons and positrons, we proceed, as in Ref. 1, according to Fock's method.[4]

In the space of second quantization the complete Hamiltonian of a system of electrons subject to an external field and interacting with the radiation field consists of three parts: the Hamiltonian of the particles in the external field, described by vector and scalar potentials $\mathbf{A}_0$ and φ_0, which is of the form (in a system of units in which $\hbar = c = 1$)

$$H_0 = \int \psi^+(x)\left[(\alpha(\mathbf{p} - e\mathbf{A}_0)) + \beta m + e\varphi_0\right]\psi(x)\,dx, \tag{1}$$

the Hamiltonian of the Coulomb interaction of the particles, which is of the form

$$H_1 = \frac{e^2}{2}\iint \psi^+(x)\,\psi(x)\frac{1}{|\mathbf{r}-\mathbf{r}'|}\psi^+(x')\,\psi(x')\,dx\,dx', \tag{2}$$

and the Hamiltonian of the interaction of the particles with the radiation field described by the vector potential $\mathbf{A}$, which is of the form

$$H' = -e\int \psi^+(x)\,(\alpha\mathbf{A})\,\psi(x)\,dx. \tag{3}$$

Here x denotes a whole set of space coordinates $\mathbf{r}$ and spinor indices. Integration with respect to x means integration with respect to $\mathbf{r}$ and summation over the spinor indices. We have chosen for the radiation field a gauge such that the scalar potential is zero, so that the Lorentz condition reduces to

$$\operatorname{div}\mathbf{A} = 0. \tag{4}$$

Later on we shall expand $\mathbf{A}$ in terms of plane waves, and the condition (4) assures the transversality of the photons. The contribution of the longitudinal and scalar photons has been explicitly separated in the form of the Coulomb interaction between the particles. The radiation field is subjected to quantization, and the external field remains unquantized.

The quantized operator Ψ is an operator for absorption of an electron or emission of a positron, and its Hermitian adjoint Ψ^+ is an operator for absorption of a positron or emission of an electron. Ψ and Ψ^+ satisfy the commutation relations

$$\{\psi_\alpha^+(\mathbf{r}), \psi_\beta(\mathbf{r}')\} = \delta_{\alpha\beta}\,\delta(\mathbf{r}-\mathbf{r}'), \quad \{\psi_\alpha^+(\mathbf{r}),\ \psi_\beta^+(\mathbf{r}')\} = \{\psi_\alpha\ (\mathbf{r}), \psi_\beta(\mathbf{r}')\} = 0, \tag{5}$$

where the spinor indices α and β take the values 1, 2, 3, 4.

The setting up of the scheme in the usual representation of the Dirac theory encounters difficulties. Since the positive-frequency and negative-frequency parts of solutions of the Dirac equation, taken separately, do not form complete systems of functions, the operators of absorption and emission of an electron, and also the operators of absorption and emission of a positron, do not satisfy commutation relations of the usual type (5). Moreover, we have a system in which there occur an infinite number of electrons of negative energy. These difficulties prevent the direct application of the Fock method. To avoid these difficulties, we proceed as is done in Ref. 1, i.e., we carry out the canonical transformations considered by Foldy and Wouthuysen.[5] After this we make the transition from the space of second quantization to configuration space. The purpose of the canonical transformations is to represent the wave functions corresponding to states with positive and negative total energy in the form of spinors in which only the upper elements or only the lower elements, respectively, are non-vanishing. Thus after the separation of the positive-frequency and negative-frequency functions the operators of absorption and emission of an electron have only the upper elements different from zero, and the absorption and emission operators of a positron only the lower elements. Then we can write the transformed quantized operators Ψ' and Ψ'^+ in the form

$$\psi' = \begin{pmatrix} \varphi \\ \chi^+ \end{pmatrix}, \quad \psi'^+ = (\varphi^+\chi), \tag{6}$$

where the two-component functions φ, φ^+, χ, and χ^+ are respectively absorption and emission operators of electron and positron. Since canonical transformations do not change the commutation relations (5), substituting the expressions (6) into Eq. (5) we get the commutation relations for the operators φ and χ in the form:

$$\{\varphi_\alpha^+(\mathbf{r}),\ \varphi_\beta(\mathbf{r}')\} = \{\chi_\alpha^+(\mathbf{r}),\ \chi_\beta(\mathbf{r}')\} = \delta_{\alpha\beta}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{7}$$

All other anticommutators of the operators φ, φ^+, χ, and χ^+ are equal to zero.

In the case of presence of a field we carry out a series of canonical transformations

$$\psi'(x) = \exp\{i(S_1+S_2+\ldots)\}\,\psi(x), \qquad \psi'^+(x) = \psi^+(x)\exp\{-i(S_1+S_2+\ldots)\}, \tag{8}$$

each of which leads to the separation of the wave functions corresponding to states with positive and negative energies to a definite order in $1/m$ ($1/m = 3.862 \times 10^{-11}\ \mathrm{cm}^{-1}$). After the performance of a certain number, say n, of these transformations, and with the neglect of terms in the transformed Hamiltonian density of orders higher than $n-1$, the eigenfunctions of this approximate Hamiltonian density of particles in the external field, corresponding to states with positive or with negative energy, will have only upper elements or only lower elements, respectively. S_n is given by the formula

$$S_n = -\left(\frac{i\beta}{2m}\right) \times \begin{pmatrix}\text{lowest-order odd operator in the}\\ \text{transformed Hamiltonian density}\end{pmatrix}^* \tag{9}$$

As the result of the three successive transformations

$$S_1 = -\frac{i}{2m}\beta(\alpha(\mathbf{p}-e\mathbf{A}_0)),\ S_2 = -\frac{i}{2m}\beta\left(-\frac{ie}{2m}\beta(\alpha\,\mathrm{grad}\,\varphi_0)\right),\ S_3 = -\frac{i}{2m}\beta(\mathbf{p}-e\mathbf{A}_0)^2(\alpha(\mathbf{p}-e\mathbf{A}_0)) \tag{10}$$

we get, expressing Ψ and Ψ^+ in terms of φ, φ^+, χ, and χ^+ and changing the order of χ and χ^+ to eliminate an infinite background energy (using $\chi\chi^+ \to -\chi^+\chi$ and $\int \chi \mathbf{p} \chi^+ dx \to \int \chi^+ \mathbf{p} \chi dx$)

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*In Ref. 5 an odd operator is understood to mean an operator, for example α, which, acting on a four-component function, mixes the upper and lower elements. The problem of separating the positive-frequency and negative-frequency functions reduces mathematically to the problem of transforming the Hamiltonian into a form free from odd operators.

$$H_0 = \int \varphi^+ \left[m + e\varphi_0 + \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_0)^2 - \frac{e}{2m}(\sigma \operatorname{curl} \mathbf{A}_0) + \frac{e}{8m^2}\Delta\varphi_0 + \frac{e}{4m^2}(\sigma [\operatorname{grad} \varphi_0 (\mathbf{p} - e\mathbf{A}_0)]) + \ldots \right] \varphi \, dx$$
$$+ \int \chi^+ \left[m - e\varphi_0 + \frac{1}{2m}(\mathbf{p} + e\mathbf{A}_0)^2 - \frac{e}{2m}(\sigma \operatorname{curl} \mathbf{A}_0) - \frac{e}{8m^2}\Delta\varphi_0 + \frac{e}{4m^2}(\tilde{\sigma} [\operatorname{grad} \varphi_0 (\mathbf{p}_0 + e\mathbf{A}_0)]) + \ldots \right] \chi \, dx \quad (11)$$

(here $\tilde{\boldsymbol{\sigma}}$ is the transposed matrix $\boldsymbol{\sigma}$)

$$H_1 = \frac{e^2}{2} \iint \varphi^+(x) \varphi(x) \frac{1}{|\mathbf{r} - \mathbf{r}'|} \varphi^+(x') \varphi(x') \, dx \, dx' + \frac{e^2}{2} \iint \chi^+(x) \chi(x) \frac{1}{|\mathbf{r} - \mathbf{r}'|} \chi^+(x') \chi(x') \, dx \, dx'$$
$$- e^2 \iint \varphi^+(x) \varphi(x) \frac{1}{|\mathbf{r} - \mathbf{r}'|} \chi^+(x') \chi(x') \, dx dx', \quad (12)$$

$$H' = H'^{\text{even}} + H'^{\text{odd}}, \quad (13)$$

where

$$H'^{\text{even}} = \int \varphi^+ \left[-\frac{e}{m}(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0)) - \frac{e}{2m}(\sigma \operatorname{curl} \mathbf{A}) + \right.$$
$$\left. + \frac{ie^2}{2m^2}(\sigma [\operatorname{grad} \varphi_0 \mathbf{A}]) + \ldots \right] \varphi dx + \int \chi^+ \left[-\frac{e}{m}(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0)) - \frac{e}{2m}(\tilde{\sigma} \operatorname{curl} \mathbf{A}) \right.$$
$$\left. - \frac{ie^2}{2m^2}(\tilde{\sigma} [\operatorname{grad} \varphi_0 \mathbf{A}]) + \ldots \right] \chi dx$$

and

$$H'^{\text{odd}} = \int \varphi^+ \mathscr{H}'^{\text{odd}} \chi^+ dx + \int \chi \mathscr{H}'^{\text{odd}} \varphi \, dx. \quad (14)$$

In Eq. (14) $\mathscr{H}'^{\text{odd}}$ has the form

$$\mathscr{H}'^{\text{odd}} = -e(\sigma \mathbf{A}) + \frac{e}{2m^2}(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0))(\sigma(\mathbf{p} - e\mathbf{A}_0)) - \frac{ie}{4m}(\sigma \nabla^*)(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0))$$
$$- \frac{ie^2}{4m}(\sigma [\operatorname{curl} \mathbf{A}_0 \mathbf{A}]) + \frac{e}{8m^2}(\sigma \operatorname{curl} \operatorname{curl} \mathbf{A}) + \frac{e}{4m^2}(\operatorname{curl} \mathbf{A}(\mathbf{p} - e\mathbf{A}_0)) + \ldots, \quad (15)$$

where ∇^* means the operator ∇ acting on $\mathbf{A}$ only.

Let us carry out the further canonical transformation, considered in Ref. 1, to get rid of the transposed matrix $\tilde{\boldsymbol{\sigma}}$:

$$\chi' = \varepsilon\chi, \quad \chi'^+ = \chi^+\varepsilon^+, \quad (16)$$

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^+ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon\varepsilon^+ = 1.$$

Using the fact that $\epsilon \tilde{\boldsymbol{\sigma}} \epsilon^+ = -\boldsymbol{\sigma}$, and dropping the primes on the transformed χ' and χ'^+, we get

$$H_0 = \int \varphi^+ \mathscr{H}_e \varphi \, dx + \int \chi^+ \mathscr{H}_p \chi \, dx,$$

where

$$\mathscr{H}_e = m + e\varphi_0 + \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_0)^2 - \frac{e}{2m}(\sigma \operatorname{curl} \mathbf{A}_0) + \frac{e}{8m^2}\Delta\varphi_0 + \frac{e}{4m^2}(\sigma [\operatorname{grad} \varphi_0 (\mathbf{p} - e\mathbf{A}_0)]) + \ldots, \quad (17)$$

$$\mathscr{H}_p = m - e\varphi_0 + \frac{1}{2m}(\mathbf{p} + e\mathbf{A}_0)^2 + \frac{e}{2m}(\sigma \operatorname{curl} \mathbf{A}_0) - \frac{e}{8m^2}\Delta\varphi_0 - \frac{e}{4m^2}(\sigma [\operatorname{grad} \varphi_0 (\mathbf{p} + e\mathbf{A}_0)]) + \ldots, \quad (18)$$

$$H'^{\text{even}} = \int \varphi^+ \left[-\frac{e}{m}(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0)) - \frac{e}{2m}(\sigma \operatorname{curl} \mathbf{A}) + \frac{ie^2}{2m^2}(\sigma [\operatorname{grad} \varphi_0 \mathbf{A}]) + \ldots \right] \varphi \, dx + \int \chi^+ \left[-\frac{e}{m}(\mathbf{A}(\mathbf{p} + e\mathbf{A}_0)) + \frac{e}{2m}(\sigma \operatorname{curl} \mathbf{A}) \right.$$
$$\left. + \frac{ie^2}{2m^2}(\sigma [\operatorname{grad} \varphi_0 \mathbf{A}]) + \ldots \right] \chi \, dx; \quad H'^{\text{odd}} = \int \varphi^+ H'^{\text{odd}} \varepsilon^+ \chi^+ dx + \int \chi \varepsilon H'^{\text{odd}} \varphi dx. \quad (19)$$

The Coulomb interaction Hamiltonian H_1 given by Eq. (12) does not change its form under the transformation (16); the commutation relations (7) also remain unchanged.

Thus we have gone over to a new representation in which positrons are regarded as particles with positive rest mass and positive charge and the operators φ, φ^+, χ, and χ^+ satisfy the commutation relations (7). This provides the necessary conditions for the direct application of the Fock method. The interaction Hamiltonian of the particles and the radiation field falls into two parts. The even part leads to transitions without change of the number of electrons and positrons, and the odd part leads to transitions with change of the number of electrons and positrons, i.e., to the annihilation and creation of pairs.

The transformed Hamiltonian of the system can be used for finding stationary states and for the calculation of probabilities of radiative transitions in cases in which the energy of the photon taking part in the process is much less than the rest energy of the electron. In such cases the theory can include the effects of certain relativistic corrections, by consideration of the higher approximations for H_0 and H'^{even} in powers of $1/m$.

In a relativistic process, in particular in annihilation, the energy of the photon is comparable with the rest energy of the electron: $k \sim m$. Since we shall expand $\mathbf{A}$ in plane waves $e^{i(\mathbf{kr})}$, each differential operator acting on $\mathbf{A}$ gives a factor k. Consequently, terms of different orders in $1/m$ can turn out to be of the same order in p/m. For example, in H'^{odd}, Eq. (15), the term $(e/8m^2)(\sigma \text{ curl curl } \mathbf{A})$ is much smaller than the term $-e(\sigma\mathbf{A})$ when $k \ll m$, but they are comparable when $k \sim m$. From this it follows that a theory restricted to an approximation of finite order in $1/m$ is applicable only to nonrelativistic processes, for which the condition $k \ll m$ holds (for example to the emission of photons produced in transitions of atomic electrons).

In order to extend the theory of relativistic processes, it is necessary to: (1) collect the terms belonging to each particular order in p/m after each canonical transformation, (2) find expressions for all the canonical transformations giving the contributions of a given order in $1/m$, and (3) find the expression for the interaction Hamiltonian H' resulting from such transformations. We confine ourselves to the accuracy of the zeroth order in p/m. The result of the transformation given by S_1 is to transform the original interaction Hamiltonian density $\mathcal{H}'^{(0)} = -e\boldsymbol{\alpha}\mathbf{A}$ as follows

$$\mathcal{H}^{(1)} = e^{iS_1}\mathcal{H}'^{(0)}e^{-iS_1} = -e\Big[(\alpha\mathbf{A}) + \frac{1}{2m}\beta\{(\alpha(\mathbf{p}-e\mathbf{A}_0)),(\alpha\mathbf{A})\} - \frac{1}{8m^2}\{(\alpha(\mathbf{p}-e\mathbf{A}_0)),\{(\alpha(\mathbf{p}-e\mathbf{A}_0)),(\alpha\mathbf{A})\}\} - \ldots\Big]. \quad (20)$$

The value of the coefficient of the n-fold anticommutator is given by the expression $\beta^n/n!(2m)^n$. We note the following property of the anticommutator of any operator containing A with the operator $(1/m)(\alpha(\mathbf{p} - e\mathbf{A}_0))$: this anticommutator gives terms that preserve the same order of smallness in p/m as that of the original operator, when in rearranging the operators one lets $\mathbf{p}$ act on $\mathbf{A}$. The other terms are of order of smallness one higher. This enables us to separate out the terms of a prescribed order, since the coefficients of the anticommutators are known.

By means of the formulas

$$\{(\alpha(\mathbf{p}-e\mathbf{A}_0)),(\alpha\mathbf{A})\} = (\sigma\text{curl}\mathbf{A}), \quad \{(\alpha(\mathbf{p}-e\mathbf{A}_0)),(\sigma\text{curl}\mathbf{A})\} = (\alpha\,\text{curl curl}\,\mathbf{A}) \quad (21)$$

valid in zeroth order in p/m, we can get from Eq. (20) the transformed interaction Hamiltonian density:

$$\mathcal{H}'^{(1)} = \mathcal{H}'^{(1)\,\text{odd}} + \mathcal{H}'^{(1)\,\text{even}},$$

where

$$\mathcal{H}'^{(1)\,\text{odd}} = -e\Big((\alpha\mathbf{A}) - \frac{1}{8m^2}(\alpha \text{ curl curl } \mathbf{A}) + \frac{1}{384\,m^4}(\alpha\,\text{curl}^{(4)}\mathbf{A}) \cdot\cdot \; \ldots + (-1)^n\frac{1}{(2n)!\,(2m)^{2n}}(\alpha\text{curl}^{(2n)}\mathbf{A}) + \ldots\Big), \quad (22)$$

$$\mathcal{H}'^{(1)\,\text{even}} = -\frac{e}{2m}\beta\Big(\sigma,\Big(\text{curl}\mathbf{A} - \frac{1}{24\,m^2}\text{curl}^{(3)}\mathbf{A} + \ldots + (-1)^n\frac{1}{(2n+1)!\,(2m)^{2n+1}}\text{curl}^{(2n+1)}\mathbf{A} + \ldots\Big)\Big), \quad (23)$$

where $\text{curl}^{(n)}\mathbf{A}$ means the result of letting the operator curl act n times on the vector $\mathbf{A}$. By means of the Lorentz condition for the radiation field, div $\mathbf{A} = 0$, we get

$$\text{curl}^{(2)}\mathbf{A} = -\Delta\mathbf{A} = k^2\mathbf{A}, \quad (24)$$

and in general $\text{curl}^{(2n)}\mathbf{A} = k^{2n}\mathbf{A}$, since $\mathbf{A}$ is written in the form of a plane wave. Using Eq. (24), we get a simplified form for $\mathcal{H}'^{(1)}$ in the zeroth approximation in p/m.

$$\mathcal{H}'^{(1)} = -e\Big(c_1^{(1)}(\alpha\mathbf{A}) + c_2^{(1)}\frac{1}{2m}\beta(\sigma\text{curl}\mathbf{A})\Big), \quad (25)$$

where

$$c_1^{(1)} = 1 - \frac{k^2}{8m^2} + \ldots + (-1)^n\frac{k^{2n}}{(2n)!\,(2m)^{2n}} + \ldots = \cos\frac{k}{2m}, \quad c_2^{(1)} = 1 - \frac{k^2}{24m^2} + \ldots + (-1)^n\frac{k^{2n}}{(2n+1)!\,(2m)^{2n}} + \ldots = \frac{2m}{k}\sin\frac{k}{2m}$$

Let us see which of the canonical transformations give a contribution in zeroth order. Obviously these will be only those that contain terms with the same number of operators p as the degree of m in the denominator. By calculation one can verify that these are the transformations with the operators S_{2n+1}, $n = 0, \ldots, \infty$. In fact, on one hand these transformations make the coefficients of even powers of the operator $\boldsymbol{\alpha}(\mathbf{p}-e\mathbf{A}_0)$ in $\mathcal{H}_0$ equal to the coefficients of the expansion of the radical $\{m^2+(\mathbf{p}-e\mathbf{A}_0)^2\}^{1/2}$ in powers of $1/m$, and on the other hand they make the coefficients of the odd powers of the operator $\boldsymbol{\alpha}(\mathbf{p}-e\mathbf{A}_0)$ equal to zero. The general expression for the part of S_{2n+1} that gives the zeroth order contribution is given by the formula

$$S'_{2n+1} = -\frac{i}{2m}\beta\cdot\left((-1)^n\frac{1}{(2n+1)m^{2n}}p^{2n}(\alpha\mathbf{p})\right). \tag{26}$$

We note that Eq. (26) can be found in another way. The transformation

$$e^{iS} = [(1+i\varkappa)/(1-i\varkappa)]^{1/4}, \quad \varkappa = -\frac{i}{m}\beta(\alpha\ (\mathbf{p}-e\mathbf{A}_0)) \tag{27}$$

found by Fock enables us to separate the solutions of the Dirac equation with positive and negative energies:

$$e^{iS}[(\alpha\ (\mathbf{p}-e\mathbf{A}_0))+\beta m]e^{-iS} = \beta\sqrt{m^2+(\mathbf{p}-e\mathbf{A}_0)^2}.$$

Consequently S at once contains the principal terms of the canonical transformations, those that give the zeroth order contributions. Regarding κ as a small quantity, one can verify that

$$e^{iS} = \exp\left\{\frac{i\varkappa}{2}+\frac{(i\varkappa)^3}{6}+\ldots+\frac{(i\varkappa)^{2n+1}}{2(2n+1)}+\ldots+\right\}.$$

Consequently, we can write

$$e^{iS} = e^{iS_1+iS_3+\ldots+iS_{2n+1}+\ldots},$$

where

$$S_{2n+1} = -\frac{i}{2m}\beta(-1)^n\frac{1}{(2n+1)m^{2n}}(\alpha(\mathbf{p}-e\mathbf{A}_0))^{2n+1}. \tag{28}$$

From Eq. (28) it follows that the part giving the zeroth order contribution is

$$S'_{2n+1} = -\frac{i}{2m}\beta(-1)^n\frac{1}{(2n+1)m^{2n}}p^{2n}(\alpha\mathbf{p}), \tag{29}$$

which agrees exactly with Eq. (26).

Let us go on to find the limiting expression for the interaction Hamiltonian H′ that results from the infinite succession of canonical transformations (26), in accuracy to the zeroth order. The original expression $\mathcal{H}'^{(1)} = -e(c_1^{(1)}(\boldsymbol{\alpha}\mathbf{A})+c_2^{(1)}(1/2m)(\boldsymbol{\sigma}\ \mathrm{curl}\ \mathbf{A}))$ is transformed in a "closed" manner, i.e., in all the transformations with the operators S_{2n+1} $(n = 1, \ldots, \infty)$ only the coefficients c_1 and c_2 are changed. This can already be seen from Eq. (26), since the operator $p^{2n} = (-1)^n\Delta^n$ leads only to the appearance of a factor k^{2n}, while commutation or anticommutation of the two parts of $\mathcal{H}'^{(1)}$ with $\boldsymbol{\alpha}\mathbf{p}$ leads only to a change of the coefficients $c_1^{(1)}$ and $c_2^{(2)}$, and so on.

Let us set

$$\Pi^{(2n+1)} = (-1)^n\frac{1}{(2n+1)m^{2n}}p^{2n}(\alpha\mathbf{p}), \qquad O^{(2n+1)} = -ec_1^{(2n+1)}(\alpha\mathbf{A}) \qquad (\text{odd part of } \mathcal{H}'^{(2n+1)}), \tag{30}$$

$$\mathscr{E}^{(2n+1)} = -ec_2^{(2n+1)}\frac{1}{2m}\beta(\sigma\,\mathrm{curl}\mathbf{A}) \qquad (\text{even part ot } \mathcal{H}'^{(2n+1)}),$$

so that $\mathcal{H}'^{(2n+1)} = O^{(2n+1)} + \mathcal{E}^{(2n+1)}$.

$\mathcal{H}'$ transforms according to the formula

$$\begin{aligned}\mathcal{H}'^{(2n+1)} = \mathcal{H}'^{(2n+1)} &+ \frac{1}{2m}\beta[\Pi^{(2n+1)}, \mathscr{E}^{(2n-1)}] - \frac{1}{8m^2}[\Pi^{(2n+1)}, [\Pi^{(2n+1)}, \mathscr{E}^{2n-1)}]] - \ldots \\ &\ldots + \frac{1}{2m}\beta\{\Pi^{(2n+1)}, O^{(2n-1)}\} - \frac{1}{8m^2}\{\Pi^{(2n+1)}, \{\Pi^{(2n+1)}, O^{(2n-1)}\}\} - \ldots.\end{aligned} \tag{31}$$

By means of the following formulas (valid to the zeroth order in p/m):

2m-fold anticommutator of $(\alpha \mathbf{A})$ and $p^{2n}(\alpha p)$

$$k^{4mn+2m}(\alpha \mathbf{A}),$$

(2m + 1)-fold anticommutator of $(\alpha \mathbf{A})$ and $p^{2n}(\alpha p)$

$$k^{2(2m+1)n+2m}(\sigma \operatorname{curl} \mathbf{A}), \tag{32}$$

2m-fold commutator of $\beta(\sigma \operatorname{curl} \mathbf{A})$ and $p^{2n}(\alpha p)$

$$k^{4mn+2m}\beta(\sigma \operatorname{curl} \mathbf{A}),$$

(2m + 1)-fold commutator of $\beta(\sigma \operatorname{curl} \mathbf{A})$ and $p^{2n}(\alpha p)$

$$-k^{2(2m+1)n+2(m+1)}\beta(\alpha \mathbf{A}),$$

we get

$$\mathscr{H}'^{(2n+1)} = -e\left(c_1^{(2n+1)}(\alpha \mathbf{A}) + c_2^{(2n+1)}\frac{1}{2m}\beta(\sigma \operatorname{curl} \mathbf{A})\right), \tag{33}$$

where the successive coefficients are connected by the formulas

$$c_1^{(2n+1)} = c_1^{(2n-1)}d_1^{(2n+1)} + c_2^{(2n-1)}d_2^{(2n+1)}, \quad c_2^{(2n+1)} = c_2^{(2n-1)}d_1^{(2n+1)} + c_1^{(2n-1)}d_3^{(2n+1)}. \tag{34}$$

Here

$$d_1^{(2n+1)} = \sum_{l=0}^{\infty}(-1)^l \frac{1}{(2l)!\,(2)^{2l}}\left(\frac{1}{2n+1}\right)^{2l}\left(\frac{k}{m}\right)^{4ln+2l} = \cos\left((-1)^n \frac{1}{2(2n+1)}\left(\frac{k}{m}\right)^{2n+1}\right),$$

$$d_2^{(2n+1)} = \sum_{l=0}^{\infty}(-1)^l \frac{1}{(2l+1)!\,(2)^{2l}}\left(\frac{1}{2n+1}\right)^{2l+1}\left(\frac{k}{m}\right)^{2(2l+1)n+2l} = -\frac{k}{2m}\sin\left((-1)^n \frac{1}{2(2n+1)}\left(\frac{k}{m}\right)^{2n+1}\right), \tag{35}$$

$$d_3^{(2n+1)} = -\frac{1}{4}\left(\frac{k}{m}\right)^2 d_2^{(2n+1)} = \frac{2m}{k}\sin\left((-1)^n \frac{1}{2(2n+1)}\cdot\left(\frac{k}{m}\right)^{2n+1}\right).$$

From this it follows that

$$c_1^{(2n+1)} = \cos\left(\frac{k}{2m}\sum_{l=0}^{2n}(-1)^l \frac{1}{2l+1}\left(\frac{k}{m}\right)^{2l}\right), \quad c_2^{(2n+1)} = \frac{2m}{k}\sin\left(\frac{k}{2m}\sum_{l=0}^{2n}(-1)^l \frac{1}{2l+1}\left(\frac{k}{m}\right)^{2l}\right). \tag{36}$$

We can find the limiting expressions $c_1^{(\infty)}$ and $c_2^{(\infty)}$, using

$$\sum_{l=0}^{\infty}(-1)^l \frac{1}{2l+1}\left(\frac{k}{m}\right)^{2l+1} = \tan^{-1}\frac{k}{m} \quad \text{for} \quad k \leqslant m, \quad c_1^{(\infty)} = \cos\left(\frac{1}{2}\tan^{-1}\frac{k}{m}\right), \quad c_2^{(\infty)} = \frac{2m}{k}\sin\left(\frac{1}{2}\tan^{-1}\frac{k}{m}\right). \tag{37}$$

For the case $k = m$

$$c_1^{(\infty)} = \cos\frac{\pi}{8}, \quad c_2^{(\infty)} = 2\sin\frac{\pi}{8}, \quad c_1^{(\infty)}c_2^{(\infty)} = \frac{1}{\sqrt{2}}. \tag{38}$$

The transformation (16) brings the interaction Hamiltonian of the particles and the radiation field into the form

$$H' = H'^{\text{even}} + H'^{\text{odd}},$$

where

$$H'^{\text{even}} = -\frac{c_2^{(\infty)}e}{2m}\left[\int \varphi^+(\sigma \operatorname{curl} \mathbf{A})\varphi\, dx + \int \chi^+(-\sigma \operatorname{curl} \mathbf{A})\chi\, dx\right], \quad H'^{\text{odd}} = -c_1^{(\infty)}e\left[\int \varphi^+(\sigma \mathbf{A}\varepsilon^+\chi^+)\, dx + \int \chi\varepsilon(\sigma \mathbf{A})\varphi\, dx\right], \tag{39}$$

expressed to zeroth order in p/m. Consequently we can apply to some relativistic processes of quantum electrodynamics $(k \sim m)$ the theory with the interaction Hamiltonian of particles and radiation field expressed by Eq. (39). In this case the theory is nonrelativistic as regards the momenta of the particles.

Let us consider a system of interacting electrons and positrons in an external field. For the study of the stationary states of the system we shall not include the interaction of the particles with the radiation field, which, as a perturbation, causes transitions between the stationary states of the system. According to the Fock method[4] the change from the space of second quantization to the configuration space is accomplished by writing the state vector of the system as a column whose elements are wave functions in the configuration space with definite numbers of particles. In all the states of occupation the difference of the numbers of electrons and positrons has a single fixed value. This is a consequence of the law of conservation of charge. The action of the quantum operators on this column must be consistent with the commutation relations.

Let us write the configuration wave function of a system of n electrons and m positrons in the form of a product of determinants of one-electron and one-positron functions, satisfying the requirements of orthogonality and normalization:

$$\psi_{\alpha_1\cdots\alpha_n,\,\beta_1\cdots\beta m}(\mathbf{r}_1\cdots\mathbf{r}_n,\,\mathbf{s}_1\cdots\mathbf{s}_m)=(n!\,m!)^{-1/2}\,\|\varphi_i(\mathbf{r}_j,\alpha_j)\|_{i,\,j=1\cdots n}\,\|\chi_i(\mathbf{s}_j,\beta_j)\|_{i,\,j=1\cdots m} \tag{40}$$

Then the variation principle $\delta W = 0$, where W is the mathematical expectation of the operator $H_0 + H_1$ in a state having n electrons and m positrons, leads, under the conditions of orthogonality and normalization of the one-electron and one-positron functions, to the Fock equations of the self-consistent field for the system of electrons and positrons in the external field, with the Hamiltonian operator expressed in the form of a power series in $1/m$ (Ref. 6).

$$\begin{gathered}\Big[\mathscr{H}_e(x)+e^2\sum_{j=1}^{n}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\varphi_j^*(x')\varphi_j(x')\,dx'-e^2\sum_{j=1}^{m}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\chi_j^*(x')\chi_j(x')\,dx'\Big]\varphi_i(x)\\ -e^2\sum_{j=1}^{n}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\varphi_j^*(x')\varphi_i(x')\,dx'\,\varphi_j(x)=\sum_{j=1}^{n}\lambda_{ij}\varphi_j(x),\qquad i=1\ldots n;\\ \Big[\mathscr{H}_p(x)+e^2\sum_{j=1}^{n}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\chi_j^*(x')\chi_j(x')\,dx'=e^2\sum_{j=1}^{n}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\varphi_j^*(x')\varphi_j(x')\,dx'\Big]\chi_i(x)\\ -e^2\sum_{j=1}^{m}\int\frac{1}{|\mathbf{r}-r'|}\chi_j^*(x')\chi_i(x')\,dx'\,\chi_j(x)=\sum_{j=1}^{m}\mu_{ij}\chi_j(x),\qquad i=1\ldots m.\end{gathered} \tag{41}$$

The state of a system of electrons and positrons subject to an external field and interacting with the radiation field is described by a state vector Ω satisfying the equation

$$(H_0+H_1+H')\,\Omega=i d\Omega/dt.$$

According to perturbation theory the transition probability per unit time is given by the formula

$$w=2\pi\,|(\Omega_n,\ H'\Omega_0)|^2\rho_n, \tag{42}$$

where Ω_0 and Ω_n are the state vectors describing the initial and final states respectively. For a second-order effect

$$(\Omega_n, H'\Omega_0)=\sum_I\frac{(\Omega_n,\ H'\Omega_I)(\Omega_I,\ H'\Omega_0)}{E_0-E_I}, \tag{43}$$

where Ω_I is the state vector describing the intermediate state.

The effect of the external field on the radiative processes comes about in two ways. Since the expression for the probability of the process involves the wave function of the system, which is a solution of the equation for the stationary states, in which the potentials of the external field appear, the external field has an effect on the radiative processes through the dependence of the wave functions of the system on the potentials of the external field. The potentials of the external field also appear explicitly in the expression (18) for the interaction Hamiltonian. For radiative processes in which the photon energy is comparable with the rest energy of the electron, the potentials of the external field do not appear in the interaction Hamiltonian as terms of zeroth order in p/m [Eq. (39)]. But the effect of the Coulomb interaction of the particles comes into the radiative processes through the wave function of the system determined by the Fock equations (41).

Let us consider by Fock's method the two-photon annihilation of a positron in a many-electron system. We assume that in the initial state there are n electrons and one positron. In the intermediate state the numbers of electrons and positrons remain the same as in the initial state, while there has been emitted a photon with wave vector $\mathbf{k}_1$ and polarization vector $\mathbf{e}_1$ (denoted as state I_1) or else $\mathbf{k}_2$, $\mathbf{e}_2$ (denoted as state II). In the final state there are two photons $\mathbf{k}_1$, $\mathbf{e}_1$ and $\mathbf{k}_2$, $\mathbf{e}_2$ and $n - 1$ electrons. The configuration wave functions of the initial, intermediate, and final electron-positron states are denoted by $\psi_{\alpha_1 \ldots \alpha, \beta_1}(\mathbf{r}_1 \ldots \mathbf{r}_n, \mathbf{s}_1)$, $\psi''_{\alpha_1 \ldots \alpha_n, \beta_1}(\mathbf{r}_1 \ldots \mathbf{r}_n, \mathbf{s}_1)$, and $\psi'_{\alpha_1 \ldots \alpha_{n-1}}(\mathbf{r}_1 \ldots \mathbf{r}_{n-1})$, respectively. Using the rules for the action of the matrix elements of quantized operators on the configuration wave functions, determined in the same way as in Ref. 4, we get[6]

$$(\Omega_{I_1}, H'\Omega_0) = -e\sqrt{\frac{2\pi}{k_1}}\int \psi''^{*}_{\alpha_1 \ldots \alpha_n, \beta_1}(\mathbf{r}_1 \ldots \mathbf{r}_n, \mathbf{s}_1)\, e^{-i(\mathbf{k}_1\mathbf{r})} \sum_{i=1}^{n} \delta_{\alpha\alpha_i}(\mathbf{r}_1 - \mathbf{r}_1) \left(-\frac{ic_2^{(\infty)}}{2m}(\sigma[\mathbf{k}_1\mathbf{e}_1])\right)_{\alpha\beta}$$

$$\times\, \psi_{\alpha_1 \ldots \alpha_{i-1}\beta\alpha_{i+1} \ldots \alpha_n, \beta_1}(\mathbf{r}_1 \ldots \mathbf{r}_{i-1}\mathbf{r}\mathbf{r}_{i+1} \ldots \mathbf{r}_n, \mathbf{s}_1)\, d\mathbf{r}_1 \ldots\ldots d\mathbf{r}_n\, d\mathbf{r}\, d\mathbf{s}_1 \tag{44}$$

$$+ (-e)\sqrt{\frac{2\pi}{k_1}}\int \psi^{*''}_{\alpha_1 \ldots \alpha_n, \beta_1}(\mathbf{r}_1 \ldots \mathbf{r}_n, \mathbf{s}_1)\, e^{-i(\mathbf{k}_1\mathbf{r})}\, \delta_{\alpha\beta_1}\, \delta(\mathbf{r} - \mathbf{s}_1) \left(\frac{ic_2^{(\infty)}}{2m}(\sigma[k_1\mathbf{e}_1])\right)_{\alpha\beta} \psi_{\alpha_1 \ldots \alpha_n, \beta}(\mathbf{r}_1 \ldots \mathbf{r}_n, \mathbf{r})\, d\mathbf{r}_1 \ldots d\mathbf{r}_n\, d\mathbf{s}_1\, d\mathbf{r},$$

$$(\Omega_n, H'\Omega_{I_1}) = (-e)\sqrt{\frac{2\pi}{k_2}}\sqrt{n}\int \psi'^{*}_{\alpha_1 \ldots \alpha_{n-1}}(\mathbf{r}_1 \ldots \mathbf{r}_{n-1})\, e^{-i(\mathbf{k}_2\mathbf{r})}\, \delta(\mathbf{r} - \mathbf{r}')\, (c_1^{(\infty)}\, \varepsilon\, (\sigma\mathbf{e}_2))_{\alpha\beta}$$

$$\times\, \psi''_{\alpha_1 \ldots \alpha_{n-1}\beta, \alpha}(\mathbf{r}_1 \ldots \mathbf{r}_{n-1}\mathbf{r}, \mathbf{r}')\, d\mathbf{r}_1 \ldots\ldots d\mathbf{r}_{n-1}\, d\mathbf{r}\, d\mathbf{r}'. \tag{45}$$

Here we have used the well known expressions for the matrix elements of the operator A (see, for example, Ref. 7). The first term in Eq. (44) consists in turn of a sum of n terms, in which the i-th term corresponds to the emission of the photon $\mathbf{k}_1$, $\mathbf{e}_1$ by the i-th electron. The second term in Eq. (44) corresponds to the emission of the photon $\mathbf{k}_1$, $\mathbf{e}_1$ by the positron. As intermediate states we must take all possible function $\psi''_{\alpha_1 \ldots \alpha_n, \beta_1}(\mathbf{r}_1 \ldots \mathbf{r}_n, \mathbf{s}_1)$. We assume that the ψ'' form a complete set:

$$\sum \psi''^{*}_{\alpha_1 \ldots \alpha_n \beta_1}(\mathbf{r}_1 \ldots \mathbf{r}_n, \mathbf{s}_1)\, \psi''_{\alpha'_1 \ldots \alpha'_n \beta_1}(\mathbf{r}'_1 \ldots \mathbf{r}'_n, \mathbf{s}'_1) = \delta_{\alpha_1\alpha'_1} \ldots \delta_{\alpha_n\alpha'_n} \ldots \delta_{\beta_1\beta'_1}\, \delta(\mathbf{r}_1 - \mathbf{r}_1) \ldots \delta(\mathbf{r}_n - \mathbf{r}'_n)\, \delta(\mathbf{s}_1 - \mathbf{s}'_1), \tag{46}$$

where the sum is taken over all possible states ψ''. We assume that initially all electrons are in bound states, and that the positron is either in a bound state or in a quasifree state with energy close to m (a "slow" positron). The matrix element (44) is appreciably different from zero only when in the intermediate state the particle that has emitted the photon $\mathbf{k}_1$ is in a quasifree state with energy close to $(m^2 + k_1^2)^{1/2}$ and each of the other particles has an energy close to m. This is so owing to the fact that in Eq. (44) the factor $e^{-i(\mathbf{kr})}$ is a very rapidly oscillating function. Consequently we take for the difference of the energies of the initial and intermediate states simply:

$$\Delta E_1 = E_0 - E_{I_1} = -\sqrt{m^2 + k_1^2} - k_1 + m, \quad \Delta E_2 = E_0 - E_{I_2} = -\sqrt{m_1^2 + k_2^2} - k_2 + m. \tag{47}$$

Using Eq. (46), we get

$$(\Omega_n, H'\Omega_0) = \sum_{I_1} \frac{(\Omega_n, H'\Omega_{I_1})(\Omega_{I_1}, H'\Omega_0)}{\Delta E_1} + \sum_{I_2} \frac{(\Omega_n, H'\Omega_{I_2})(\Omega_{I_2}, H'\Omega_0)}{\Delta E_2} = e^2 \frac{2\pi}{\sqrt{k_1 k_2}} \frac{\sqrt{n}}{\Delta E_1} \int \psi'^{*}_{\alpha_1 \ldots \alpha_{n-1}}(\mathbf{r}_1 \ldots \mathbf{r}_{n-1})$$

$$\times \Bigl[\sum_{i=1}^{n} e^{-i(\mathbf{k}_2\mathbf{r}_n)} e^{-i(\mathbf{k}_1\mathbf{r}_i)}\, \delta(\mathbf{r}_n - \mathbf{s}_1)\, (c_1^{(\infty)}\, e\, (\sigma\mathbf{e}_2))_{\beta_1\alpha_n} \left(-\frac{ic_2^{(\infty)}}{2m}(\sigma[\mathbf{k}_1\mathbf{e}_1])\right)_{\alpha_i\beta} \psi_{\alpha_1 \ldots \alpha_{i-1}\beta\alpha_{i+1} \ldots \alpha_n, \beta_1}(\mathbf{r}_1 \ldots \mathbf{r}_n, \mathbf{s}_1)$$

$$+ e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{s}_1)}\, \delta(\mathbf{r}_n - \mathbf{s}_1)\, (c_1^{(\infty)}\, \varepsilon\, (\sigma\mathbf{e}_2))_{\alpha_n\beta_1} \left(\frac{ic_2^{(\infty)}}{2m}(\sigma[\mathbf{k}_1\mathbf{e}_1])\right)_{\beta_1\beta} \psi_{\alpha_1 \ldots \alpha_n, \beta}(\mathbf{r}_1 \ldots \mathbf{r}_n, \mathbf{s}_1)\Bigr]\, d\mathbf{r}_1 \ldots d\mathbf{r}_n\, d\mathbf{s}_1$$

$$\left(\begin{array}{l}\text{corresponding term obtained by interchanging} \\ \Delta E_1,\ \mathbf{k}_1,\ \text{and}\ \mathbf{e}_1\ \text{with}\ \Delta E_2,\ \mathbf{k}_2,\ \text{and}\ \mathbf{e}_2,\ \text{respectively}\end{array}\right) \tag{48}$$

The sum $\sum_{i=1}^{n}$ in Eq. (48) consists of two parts, namely the term with $i = n$ and the terms with $i \neq n$.

The terms with $i \neq n$ make a negligibly small contribution to the matrix element because of the smallness of the integrals over $\mathbf{r}_i$ and $\mathbf{r}_n$ of the rapidly oscillating functions $e^{-i(\mathbf{k}_2\mathbf{r}_n)}$ and $e^{-i(\mathbf{k}_1\mathbf{r}_i)}$. Physically this corresponds to the fact that in the transition from the initial state to the intermediate state the i-th electron emits the photon $\mathbf{k}_1$ and the momentum $-\mathbf{k}_1$ is transferred to the nucleus, and in the transition from the intermediate to the final state the n-th electron is annihilated with the positron $(n \neq i)$ with the emission of the photon $\mathbf{k}_2$ and the momentum $-\mathbf{k}_2$ is transferred to the nucleus. Such a process, in which there twice occurs transfer of momentum comparable with m, is extremely improbable. As for the terms with $i = n$, they make an important contribution to the matrix element when $|\mathbf{k}_1 + \mathbf{k}_2|$ is small. When $|\mathbf{k}_1 + \mathbf{k}_2|$ is small we can replace $\mathbf{k}_1$ and $\mathbf{k}_2$ by $\mathbf{k}$, which, by the law of conservation of energy, is equal to m, and ΔE_1 and ΔE_2 by $-(2m)^{1/2}$. Neglecting the terms with $i \neq n$ and setting

$$K = -\frac{i}{2\sqrt{2m}}\,[\varepsilon(\sigma\mathbf{e}_2)(\sigma[\mathbf{k}_1\mathbf{e}_1]) + \varepsilon(\sigma\mathbf{e}_1)(\sigma[\mathbf{k}_2\mathbf{e}_2])],$$

we get

$$(\Omega_n,\ H'\Omega_0) = -e^2\frac{\sqrt{2}\pi}{km}\sqrt{n}\int \psi^{*}_{\alpha_1\ldots\alpha_{n-1}}(\mathbf{r}_1\ldots r_{n-1})\,e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_n)}(K_{\beta,\alpha_n} - K_{\alpha_n\beta_1})\cdot\psi_{\alpha_1\ldots\alpha_{n-1}\alpha_n,\,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_{n-1}\mathbf{r}_n,\ \mathbf{r}_n)\,d\mathbf{r}_1\ldots d\mathbf{r}_n. \tag{49}$$

The probability per unit time of two-photon annihilation is given by the formula

$$w = (k^2/4\pi)\,|(\Omega_n,\ H'\Omega_0)|^2. \tag{50}$$

The dependence of the probability of the process on $|\mathbf{k}_1 + \mathbf{k}_2|$ is clearly shown in Eq. (49). The process is most probable when $|\mathbf{k}_1 + \mathbf{k}_2| = 0$, i.e., the photons emitted must have their directions correlated to be antiparallel. With increase of the departure of the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$ from the value π (for $k_1 = k_2$) the probability decreases rapidly. The process is most probable when each photon has an energy equal to the rest energy of the electron: $k_1 = k_2 = m$. With increase of the difference between the individual photon energies and m, for $k_1 + k_2 = 2m$, the probability decreases rapidly. From this one can get the natural line width of the annihilation radiation. Since the pair being annihilated is bound to the nucleus and the other electrons, the law of conservation of momentum need not be exactly fulfilled, as it must be for free particles, in the sense that the total momentum of the pair annihilated must be equal to the total momentum of the emitted photons. That is, in the center-of-mass system of the electron and positron the emitted photons can have a total momentum $\mathbf{k}_1 + \mathbf{k}_2$ different from zero, and this amount of momentum is transferred to the nucleus. The larger the momentum transferred, the less probable the process.

Let us write ψ and ψ' in the form

$$\psi_{\alpha_1\ldots\alpha_n,\,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_n,\ \mathbf{s}_1) = \frac{1}{\sqrt{n!}}\,\|\varphi_i(\mathbf{r}_j,\ \alpha_j)\|_{i,j=1\ldots n}\,\chi_1(r_n,\ \beta_1), \tag{51}$$

$$\psi'_{\alpha_1\ldots\alpha_{n-1}}(\mathbf{r}_1\ldots\mathbf{r}_{n-1}) = \frac{1}{\sqrt{(n-1)!}}\,\|\varphi'_i(\mathbf{r}_j,\ \alpha_j)\|_{i,j=1\ldots n-1}. \tag{52}$$

Substituting Eqs. (51) and (52) into Eq. (49) and developing the determinant $\|\varphi_i(r_j,\ \alpha_j)\|$ in terms of the elements $\varphi_i(r_j,\ \alpha_j)$, we get

$$(\Omega_n,\ H'\Omega_0) = -e^2\frac{\sqrt{2}\pi}{km}\sqrt{n}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{(n-1)!}}\,[K_{\beta_1\alpha_n} - K_{\alpha_n^{\bullet}\beta_1}]\sum_{l=1}^{n}(-1)^{l+n}$$

$$\times\int\|\varphi_i'^{*}(\mathbf{r}_j,\ \alpha_j)\|_{i,j=1\ldots n-1}\,\|\varphi_k(\mathbf{r}_j,\ \alpha_j)\|_{\substack{k=1\ldots l-1,\ l+1\ldots n\\ j=1\ldots n-1}}\,dx_1\ldots dx_{n-1}\int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_n)}\,\varphi_l(\mathbf{r}_n,\alpha_n)\,\chi_1(r_n,\beta_1)\,d\mathbf{r}_n, \tag{53}$$

where integration over x_i means integration over $\mathbf{r}_i$ and summation over α_i, and repeated indices are summed over. The sum $\sum_{i=1}^{n}$ in Eq. (53) includes the various acts of annihilation and the effect of electronic correlation. Since φ_i and φ'_j relate respectively to the system atom + positron and to the ion that remains after the annihilation, they are not orthogonal to each other. Consequently, the possibility of transitions of electrons accompanying the annihilation is not excluded.

Let us assume that the remaining electrons have made no transitions during the annihilation, i.e., that

φ_i and φ_i' have the same quantum numbers; then in the sum $\sum_{\ell=1}^{n}$ the term with $\ell = n$ makes the principal contribution to the matrix element. The main part of this term is

$$\sum_{i_1 \ldots i_{n-1}} \int \varphi_{i_1}'^{*}(x_1)\, \varphi_{i_1}(x_1)\, dx_1 \int \varphi_{i_2}'^{*}(x_2)\, \varphi_{i_2}(x_2)\, dx_2 \ldots \int \varphi_{i_{n-1}}'^{*}(x_{n-1})\, \varphi_{i_{n-1}}(x_{n-1})\, dx_{n-1}$$

$$= (n-1)! \prod_{i=1}^{n-1} \int \varphi_i'^{*}(\boldsymbol{x})\, \varphi_i(x_i)\, dx_i, \tag{54}$$

where $i_1, \ldots, i_{n-1}$ is a permutation of the numbers $1, \ldots, n-1$, and the summation is taken over all permutations $i_1, \ldots, i_{n-1}$. The other part of the term is a correction to the main part (54), for example

$$\int \varphi_1'^{*}(x_1)\, \varphi_2(x_1)\, dx_1 \int \varphi_2'^{*}(x_1)\, \varphi_1(x_2)\, dx_2 \ldots$$

and so on. The other terms in the sum $\sum_{\ell=1}$, with $\ell \neq n$, are corrections to the main part of the term with $\ell = n$ given by Eq. (54), and correspond to annihilations in which the positron is annihilated with the electron that was in the state ℓ ($\ell \neq n$), but the annihilation is accompanied by the transition of another electron into this state. This is an effect of electronic correlation.

Let us examine the possibility of reducing the problem of the annihilation of a positron in a many-electron system to the corresponding one-electron problem. If we assume that the annihilation of the pair does not change the one-electron functions of the remaining electrons, then ψ' is given by the determinant

$$\| \varphi_i(r_j, \alpha_j) \|_{i,\, j=1 \cdots n-1} / \sqrt{(n-1)!},$$

where φ appears instead of φ'. Then the only nonvanishing contribution in the sum $\sum_{\ell=1}^{n}$ in Eq. (53) is the main part of the term with $\ell = n$, Eq. (54); the effect of electronic correlation disappears, and also no transitions of the remaining electrons are possible. The result is that Eq. (53) reduces to the following:

$$- e^2 \frac{\sqrt{2}\,\pi}{km} (K_{\beta_1 \alpha_n} - K_{\alpha_n \beta_1}) \int e^{-i((\mathbf{k}_1 + \mathbf{k}_2)\mathbf{r}_n)}\, \varphi_n(\mathbf{r}_n, \alpha_n)\, \chi_1(\mathbf{r}_n, \beta_1)\, d\mathbf{r}_n. \tag{55}$$

This is just the result obtained by multiplying the integral

$$\int e^{-i((\mathbf{k}_1 + \mathbf{k}_2)\mathbf{r}_n)}\, \varphi_n(\mathbf{r}_n)\, \chi_1(\mathbf{r}_n)\, d\mathbf{r}_n. \tag{56}$$

by the corresponding matrix element for the annihilation of free particles. Consequently, the reduction of the problem of annihilation in a many-electron system to the corresponding one-electron problem is equivalent to the assumption that the wave functions of the remaining electrons are not changed as a result of the annihilation, or the neglect of the possibility of transitions by the remaining electrons and of electronic correlation effects.

As a concrete example let us consider the two-photon annihilation of a positron in helium. The wave function of the system is of the form

$$\psi_{\alpha_1, \alpha_2 \beta_1}(\mathbf{r}_1 \mathbf{r}_2 \mathbf{s}_1) = \psi(\mathbf{r}_1 \mathbf{r}_1 \mathbf{s}_1)\, \Phi(\alpha_1 \alpha_2 \beta_1). \tag{57}$$

Let us consider separately the cases of orthohelium and parhelium. The Schrödinger function of the system orthohelium + positron has the form

$$\psi_{\text{ortho}}(\mathbf{r}_1 \mathbf{r}_2 \mathbf{s}_1) = (\varphi_1(\mathbf{r}_1)\, \varphi_2(\mathbf{r}_2) - \varphi_2(\mathbf{r}_1)\, \varphi_1(\mathbf{r}_2))\, \chi_1(\mathbf{s}_1)/\sqrt{2}, \tag{58}$$

and that of the system parhelium + positron has the form

$$\psi_{\text{para}}(\mathbf{r}_1 \mathbf{r}_2 \mathbf{s}_1) = (\varphi_1(\mathbf{r}_1)\, \varphi_2(\mathbf{r}_2) + \varphi_2(\mathbf{r}_1)\, \varphi_1(\mathbf{r}_2))\, \chi_1(\mathbf{s}_1)/\sqrt{2}. \tag{59}$$

The spin functions $\Phi(\alpha_1\alpha_2\beta_1)$ can be obtained by combining the spin functions of orthohelium and parhelium, $\chi(\alpha_1\alpha_2)$, with the spin functions $S(\beta_1)$ of the positron, and the resulting functions must be eigenfunctions of the total spin of the system and of its z component, and of the total spin of the two electrons and the spin of the positron. As a result we get six spin states for the system orthohelium + positron:

$$\Phi_{3/2}^{3/2} = \chi_{11} S_{1/2},\ \Phi_{3/2}^{-3/2} = \chi_{1\,-1} S_{-1/2},\ \Phi_{3/2}^{1/2} = \sqrt{\frac{1}{3}}\,\chi_{11} S_{-1/2} + \sqrt{\frac{2}{3}}\,\chi_{10} S_{1/2},\ \Phi_{3/2}^{-1/2} = \sqrt{\frac{2}{3}}\,\chi_{10} S_{-1/2} + \sqrt{\frac{1}{3}}\,\chi_{1\,-1} S_{1/2},$$

$$\Phi_{1/2}^{1/2} = \sqrt{\frac{2}{3}}\,\chi_{11} S_{-1/2} - \sqrt{\frac{1}{3}}\,\chi_{10} S_{1/2},\ \Phi_{1/2}^{-1/2} = \sqrt{\frac{1}{3}}\,\chi_{10} S_{-1/2} - \sqrt{\frac{2}{3}}\,\chi_{1\,-1} S_{1/2}; \tag{60}$$

and two spin states for the system parhelium + positron:

$$\Phi_{1/2}^{1/2} = \chi_{00} S_{1/2}, \quad \Phi_{1/2}^{-1/2} = \chi_{00} S_{-1/2}. \tag{61}$$

The lower index on the Φ denotes the total spin of the system, and the upper index gives its z component; the indices on χ are the total spin of the two electrons and its z component; that on the S is the spin component of the positron.

Let us substitute Eqs. (58) to (61) into Eq. (53), average over all spin states of the initial state, and sum over the polarizations of the emitted photons; then we get the probability per unit time of two-photon annihilation of a slow or bound positron with the remaining electron of arbitrary spin in a state described by the wave function φ':

$$w_{\text{ortho}} = e^4 \frac{\pi}{2m^2}\left[\int \varphi'^*(\mathbf{r}_1)\,\varphi_1(\mathbf{r}_1)\,d\mathbf{r}_1 \int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_2)}\,\varphi_2(\mathbf{r}_2)\,\chi_1(\mathbf{r}_2)\,d\mathbf{r}_2 - \int \varphi'^*(\mathbf{r}_1)\,\varphi_2(\mathbf{r}_1)\,d\mathbf{r}_1 \int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_2)}\,\varphi_1(\mathbf{r}_2)\,\chi_1(\mathbf{r}_2)\,d\mathbf{r}_2\right]^2 \tag{62}$$

for the system orthohelium + positron, and

$$w_{\text{para}} = e^4 \frac{\pi}{2m^2}\left[\int \varphi'^*(\mathbf{r}_1)\,\varphi_1(\mathbf{r}_1)\,d\mathbf{r}_1 \int e^{-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_2}\,\varphi_2(\mathbf{r}_2)\,\chi_1(\mathbf{r}_2)\,d\mathbf{r}_2 + \int \varphi'^*(\mathbf{r}_1)\,\varphi_2(\mathbf{r}_1)\,d\mathbf{r}_1 \int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_2)}\,\varphi_1(\mathbf{r}_2)\,\chi_1(\mathbf{r}_2)\,d\mathbf{r}_2\right]^2 \tag{63}$$

for the system parhelium + positron.

When the positron is in a bound state (bound to the atom), one must sum up the probabilities of annihilation of the positron with the electrons in the various states, summing over the Schrödinger functions and spin functions of the final state, and also over **k**. The reciprocal of the resulting sum gives the lifetime of the positron, independent of the density of the medium. When the positron is in a quasifree state, one must take a wave function normalized to unit volume. The reciprocal of the resulting probability, obtained by summing over all final states, specified by Schrödinger and spin functions and by the wave vector **k**, and multiplying by the number of atoms per unit volume, gives the lifetime of the quasifree positron in the medium.

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MAGNETIC SUSCEPTIBILITY OF A SEMICONDUCTOR WITH AN IMPURITY BAND IN A STRONG MAGNETIC FIELD

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The magnetic susceptibility of semiconductors with impurity bands in strong magnetic fields is considered. It is established that for a given dispersion law the susceptibility has a term which oscillates with increasing field strength. Since these oscillations depend substantially on the properties of the model accepted for the semiconductor with an impurity band, their experimental observation can yield a partial confirmation of the correctness of the proposed model.

1. INTRODUCTION

CONSIDERABLE attention has been devoted recently to experimental and theoretical investigation of the properties of semiconductors with impurity bands, especially semiconductors of the Ge type.[1–3]

In spite of this, very little is known about the nature of the impurity band, the dispersion law, and the distribution of the impurity atoms in the host lattice. One way to investigate these questions is to invent a specific reasonable model for semiconductors of this type. One would then consider those parameters whose behavior depends fundamentally on the assumed model. By comparing theory with experiment, one can then determine whether the model is correct.

The parameters to be considered, probably, can not be the electrical conductivity or the Hall coefficient, whose qualitative behavior does not basically depend on the particular properties of models. It will be shown in the course of this paper, that a suitable property for investigation is the magnetic susceptibility in strong magnetic fields H.

For this purpose, we shall limit the discussion to the simplest kind of semiconductor, namely, one which has an energy spectrum that contains a conduction band and a fundamental impurity s-band (for a p-type semiconductor it would contain a valence band and an acceptor impurity band). The impurity to be considered will have only one valence and its concentration per cc will be denoted by n_0. The forbidden energy gap, $\Delta\epsilon$, between the bottom of the conduction band and "the top" of the impurity band will be considered positive.

If the impurity band is narrow, i.e., its width is $D \lesssim kT$, one must take into account how far the relation between the energy of the electron, $\epsilon(\mathbf{k})$, departs from a quadratic dependence on the wave vector. We shall assume that $\epsilon(\mathbf{k})$ is nearly periodic in the fundamental impurity s-band. That this assumption is valid will become clear as we progress. For simplicity, we shall assume that

$$\varepsilon(\mathbf{k}) = -\Delta[\cos k_x a + \cos k_y a + \cos k_z a], \tag{1}$$

i.e., it is as if the impurity atoms were distributed in a simple cubic lattice with a lattice constant, a, given by: $a = n_0^{-1/3}$. In Eq. (1) the following relations were used:

$$-\pi \leqslant k_x a,\ k_y a,\ k_z a \leqslant \pi, \quad \Delta = D/6 \tag{2}$$

and the zero-energy value was assigned to the center of the impurity band.

It is in this model that we shall now consider the magnetic susceptibility in a strong magnetic field, H.

2. MAGNETIC SUSCEPTIBILITY OF ELECTRONS IN THE IMPURITY BAND

Calculation of the magnetic susceptibility of electrons in the impurity band presupposes a solution of the corresponding statistical sum:

$$Z_H = \mathrm{Sp}\exp(-\hat{\mathscr{H}}/kT), \tag{3}$$

where to the particular degree of approximation used here[4]

$$\hat{\mathscr{H}} = -\Delta\left[\cos k_x a + \cos\left(k_y a + \frac{eH}{\hbar c}ax\right) + \cos k_z a\right] + s\mu_B H, \tag{4}$$

if the field, **H**, is applied along the Z-axis and the corresponding vector potential is chosen; $s = \pm 1$; μ_B is the Bohr magnetron, and $\Delta = \Delta(H)$. This yields a closed expression for Z_H for any non-vanishing H and $\Delta/kT = b_0$. However, one must take it into account that impurity bands, generally speaking, are narrow and their width D is less than or of the order of kT. Consequently, it is assumed that $b_0 < 1$, and we proceed to calculate Z_H under this assumption.

To calculate Z_H, it is expedient to transfer to the k-representation, i.e. to write $x = i\partial/\partial k_x$. The value of Z_H can be calculated to terms of the order of b_0^4 inclusively and for any H (in the case of an impurity band); therefore, since the part of $\hat{\mathscr{H}}$, which depends on k_z and on the spin of the electron, enters additively, one has the following expression for Z_H

$$Z_H = \frac{1}{2\pi a^3}\int_{-\pi}^{\pi} e^{b_0 \cos t}dt\,(e^{\mu_B H/kT} + e^{-\mu_B H/kT})\,\mathrm{Sp}\exp\left[b_0\cos X + b_0\cos\left(Y + i\alpha\frac{\partial}{\partial X}\right)\right],$$

$$-\pi \leqslant X = k_x a \leqslant \pi;\ -\pi \leqslant Y = k_y a \leqslant \pi;\ \alpha = eHa^2/\hbar c,$$

from which

$$Z_H = \frac{2}{a^3} I_0(b_0)\cosh\frac{\mu_B H}{kT}\,\mathrm{Sp}\exp(\hat{A} + \hat{B}), \tag{5}$$

where

$$\hat{A} = b_0\cos X,\quad \hat{B} = b_0\cos(Y + i\alpha\partial/\partial X),$$

and I_0 is Bessel's function of zero order with imaginary argument. Since in Eq. (5) the operators $\hat{A}$ and $\hat{B}$ do not commute, one can apply Feynman's theorem[5] to calculate $\mathrm{Sp}\exp(\hat{A} + \hat{B})$:

$$Z'_H = \mathrm{Sp}\exp(\hat{A} + \hat{B}) = \mathrm{Sp}\exp\hat{A}\exp\left[\int_0^1 ds\exp(-s\hat{A})\,\hat{B}\exp(s\hat{A})\right],$$

from which

$$\exp(-s\hat{A})\,\hat{B}\exp s\hat{A} = \hat{B} + \exp(-s\hat{A})\,[\hat{B}, \exp s\hat{A}].$$

Using this theorem again,

$$\mathrm{Sp}\exp(\hat{A} + \hat{B}) = \mathrm{Sp}\,e^{\hat{A}}e^{\hat{B}}\exp\left\{\int_0^1 ds e^{-s\hat{B}}\left(\int_0^1 ds' e^{-s'\hat{A}}\,[\hat{B}, e^{s'\hat{A}}]\right)e^{s\hat{B}}\right\} = \mathrm{Sp}\exp\hat{A}\exp\hat{B}\exp\left[\int_0^1 dsT(s)\right],$$

since the index s′ does not play a regulating role,[5] so that

$$T(s) = e^{-s\hat{B} - s\hat{A}}\,[\hat{B}, e^{s\hat{A}}]\,e^{s\hat{B}}.$$

If T(s) is found to be small, then one can use for Z'_H the expression

$$Z'_H = \mathrm{Sp}\exp\hat{A}\exp\hat{B}\sum_{n=0}^{\infty} W_n,\ W_0 = 1,\ W_n = \int_0^1 ds_1 \ldots \int_0^{s_{n-1}} ds_n T(s_1)\ldots T(s_n). \tag{6}$$

It should be noted, that as in the case which is under consideration, with $b_0 < 1$, $T(s) \sim b_0^2$, the series in Eq. (6) converges sufficiently rapidly, because the principal term in W_n is proportional to b_0^{2n}.

To calculate the commutator in T(s) we make use of the properties of the displacement operator $\exp(\pm iY \mp \alpha\partial/\partial X)$.

Then

$$[\hat{B}, \exp s\hat{A}] = \frac{b_0}{2}\{[e^{sb_0}\cos(X + \alpha) - e^{sb_0}\cos X]\,e^{-iY + \alpha\partial/\partial X} + [e^{sb_0}\cos(X - \alpha) - e^{sb_0}\cos X]\,e^{iY - \alpha\partial/\partial X}\}.$$

Since it is not possible to calculate Z_H' for all values of b_0, we can make use of the fact, that in our case it is possible to substitute $b_0 < 1$ and to calculate the terms in Z_H', by expanding in powers of b_0, the quantities $\exp sb_0 \cos X$ and $\exp sb_0 \cos(Y + i\alpha\partial/\partial X)$, keeping in Z_H' only terms proportional to b_0^4.

In what follows in place of $T(s)$ for simplicity we calculate the quantity

$$T_\eta(s) = \exp \eta s \hat{A} \exp s\hat{B}\,[\exp(-s\hat{B}), \hat{A}] \exp(-s\hat{A}), \tag{7}$$

where η is a certain parameter.

In order to obtain an expression for Z_H' to an accuracy including terms of b_0^4, it is necessary to calculate in Eq. (6) the quantities W_1 and W_2, since, as is evident from Eqs. (6) and (7), $W_n \sim b_0^{2n}$:

$$Z_H' = \mathrm{Sp}\,\{e^{b_0} \cos X e^{b_0} \cos(Y + i\alpha\partial/\partial X)(1 + W_1 + W_2)\}. \tag{8}$$

Subsequent calculation of $T(s)$ and $T_\eta(s)$ is straightforward although very cumbersome. Therefore, we shall not go through the details, and will give instead the result of the calculation of Z_H'. It is simple to prove that

$$\mathrm{Sp}\, e^{b_0} \cos X e^{b_0} \cos(Y + i\alpha\partial/\partial x) = I_0^2(b_0) \tag{9}$$

for any H and b_0, and it is independent of H; the dependence on H comes about because of the non-commutability of the operators $\hat{A}$ and $\hat{B}^4$. To an accuracy of terms of the order of b_0^4:

$$\mathrm{Sp}\, e^{b_0} \cos X e^{b_0} \cos(Y + i\alpha\partial/\partial X)\, W_1 = \int_0^1 ds\, \mathrm{Sp}\, e^{b_0} \cos X \cdot T_\eta(s), \tag{10}$$

in which $\eta s = s - 1$. Noting that

$$\mathrm{Sp}\, f(X) \cos(Y + i\alpha\partial/\partial X) = \mathrm{Sp}\, f(X) \cos^3(Y + i\alpha\partial/\partial X) = \ldots = 0,$$

$$\mathrm{Sp}\, f(X) \cos^2(Y + i\alpha\partial/\partial X) = \frac{1}{2}\int_{-\pi}^{\pi} f(X)\, dX,$$

we obtain that

$$\int_0^1 ds\, \mathrm{Sp}\left(1 + b_0 \cos X + \frac{b_0^2}{2}\cos^2 X\right) T_\eta(s) = \frac{b_0^4}{12}\sin^2\frac{\alpha}{2}\,. \tag{11}$$

To this degree of accuracy,

$$\mathrm{Sp}\, e^{b_0} \cos X e^{b_0} \cos\left(Y + i\alpha\frac{\partial}{\partial X}\right) W_2 = \frac{b_0^4}{8}\left(\sin^2\frac{\alpha}{2}\ \sin^2\alpha\ - \sin^6\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\cos^4\frac{\alpha}{2}\right). \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (8) and Z_H' of Eq. (5), we obtain:*

$$Z_H = \frac{2}{a^3} I_0(b_0) \cosh\frac{\mu_B H}{kT}\left\{I_0^2(b_0) + \frac{b_0^4}{24}\Phi(\alpha)\right\},\quad \Phi(\alpha) = \sin^2\frac{\alpha}{2}\left(2 - 3\cos^4\frac{\alpha}{2}\right) + 3\sin^2\frac{\alpha}{2}\sin^2\alpha - 3\sin^6\frac{\alpha}{2}. \tag{13}$$

Using the methods of contour integration[2,6] and the expression for Z_H one can obtain an expression for the density of electrons η in the impurity band:

$$\begin{aligned} n(\bar{\mu}, H, T) &= n_0\left\{1 + \frac{2}{\pi}\int_0^\infty d\beta\,\frac{\sin\bar{\mu}\beta\cos\bar{\mu}_B H\beta}{\sinh\beta}\left[I_0^3(b\beta) + \frac{b^4\beta^4}{12}\,\Phi(\alpha)\right]\right\},\quad \Delta n \equiv n(\bar{\mu}, H, T) - n(\bar{\mu}_0, 0, T) \\ &= \frac{2}{\pi}\int_0^\infty d\beta\left[\frac{I_0^3(b\beta)}{\sinh\beta}(\sin\bar{\mu}\beta\cos\bar{\mu}_B H\beta - \sin\bar{\mu}_0\beta) + \Phi(\alpha)\,\frac{b^4\beta^4\sin\bar{\mu}\beta\cos\bar{\mu}_B H\beta}{12\sinh\beta}\right], \end{aligned} \tag{14}$$

where $\bar{\mu} = \mu/\pi kT$, $b = \Delta/\pi kT$, $\bar{\mu}_0 = \bar{\mu}\,(H = 0)$. Here μ is the chemical potential, $\bar{\mu}_B = \mu/\pi kT$, and I_0 is Bessel's Function of zero order with a real argument.

Using Z_H and the methods of contour integration, one can also obtain an expression for the magnetic susceptibility of the electrons in the impurity band:[6]

$$\chi_1 = \chi_0 + \chi_{sp} + \chi_{0sp},\ \chi_0 = -2s_0\frac{\Phi'(\alpha)}{\alpha}\sum_{i=1,2}\left(1 - \tanh^2\frac{\pi\bar{\mu}_i}{2}\right)\left(1 - 2\tanh^2\frac{\pi\bar{\mu}_i}{2}\right),\ \chi_{sp} = \frac{n_0\mu_B}{H}\sum_{i=1,2}\tanh\frac{\pi\bar{\mu}_i}{2};$$

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*The expression for $Z_H'(T)$ differs from the expression for Z_H for $H = 0$ only by terms of the order of $b_0^4 \ll 1$.

$$\chi_{0\mathrm{sf}} = 2s_0\pi\bar{\mu}_{\mathrm{B}}H_0 \sum_{i=1,2} \tanh\frac{\pi\bar{\mu}_i}{2}\left(1-\tanh^2\frac{\pi\bar{\mu}_i}{2}\right)\left(3-4\tanh^2\frac{\pi\bar{\mu}_i}{2}\right),$$
$$s_0 = e^2\Delta b^3 a/768\,\hbar^2c^2,\quad \Phi'(\alpha)\equiv d\Phi(\alpha)/d\alpha;\quad \mu_{1,2}=\mu_{\mathrm{B}}H\pm\mu,\quad H_0\equiv\hbar c/ea^2; \tag{15}$$

where the bar over $\mu_{1,2}$ means division of that quantity by πkT.

3. MAGNETIC SUSCEPTIBILITY OF A SEMICONDUCTOR WITH AN IMPURITY BAND

In what follows we shall not take into account the magnetism of atomic impurities. Let us assume that the complete magnetic susceptibility is approximately equal to the sum of the susceptibility of the impurity band χ_1 and that of the electrons in the conductivity band χ_2. An expression for χ_2 can be found for any field, provided that the electron gas in the conductivity band of the semiconductor with an impurity band is non-degenerate.

Making use of the well-known expression for the statistical sum of electrons in a field H, we obtain

$$\chi_2 = \frac{kT}{H^2}N(H,T)\left\{1+\pi\bar{\mu}_{\mathrm{B}}H\tanh\pi\bar{\mu}_{\mathrm{B}}H-\frac{\hbar\omega_0}{2kT}\coth\frac{\hbar\omega_0}{2kT}\right\},$$
$$\omega_0=\frac{eH}{m^*c},\quad N(H,T)=Z_0\frac{\hbar\omega_0\cosh\pi\bar{\mu}_{\mathrm{B}}H}{2kT\sinh(\hbar\omega_0/2kT)}\exp\left(\pi\bar{\mu}-\frac{D_1}{kT}\right),\quad Z_0=2(2\pi m^*kT/h^2)^{3/2}, \tag{16}$$

where $N(H,T)$ is the equilibrium concentration of electrons in the conductivity band, μ is the chemical potential, $D_1 = 3\Delta + \Delta\epsilon$, m^* is the mean value of the effective mass of the electrons, and $\Delta\epsilon$ is the energy gap between the conductivity band and the fundamental impurity band; it is assumed that for $T \lesssim 10°$ K, $\Delta\epsilon > kT$. For larger fields, H, i.e. for $\pi\mu_{\mathrm{B}}H \gg 1$,

$$\chi_2 \approx kTH^{-2}N(H,T)\{1+\pi\bar{\mu}_{\mathrm{B}}H(1-m/m^*)\}.$$

It is easily shown, that, in the conductivity band, electron gas is non-degenerate as was assumed in deriving the expression for χ_2. To prove this, we must calculate $\mu(H,T)$. To do this, let us make use of the electrical neutrality equation:

$$n_0\left\{1+\frac{2}{\pi}\int_0^\infty d\beta\,\frac{\sin\bar{\mu}\beta\cos\bar{\mu}_{\mathrm{B}}H\beta}{\sinh\beta}\left[J_0^3(b\beta)+\frac{b^4\beta^4}{2}\Phi(\alpha)\right]\right\}+N(H,T)=n_0. \tag{17}$$

To calculate χ_1 to an accuracy of b_0^4, it is sufficient to solve for μ from Eq. (17), neglecting corrections of the order of b_0^2. With reference to $\chi_0+\chi_{0\mathrm{sp}}$, further explanation is not required since $\chi_0+\chi_{0\mathrm{sp}} \sim b_0^4$. As regards χ_{sp}, this quantity will be of interest in what follows only when it is of the same order of magnitude as $\chi_0+\chi_{0\mathrm{sp}}$. This occurs, for example, for sufficiently large values of a, since $\chi_{\mathrm{sp}} \sim 1/a^3$ and $\chi_0+\chi_{0\mathrm{sp}} \sim a$. Such a situation occurs in a semiconductor with an impurity band if $\eta_0 \sim 10^{15}-10^{18}$ so that $a \sim 10^{-5}-10^{-6}$cm.

In this case Eq. (17) becomes (for large H)

$$n_0\left(\tanh\frac{\pi\bar{\mu}_2}{2}-\tanh\frac{\pi\bar{\mu}_1}{2}\right)=N(H,T),$$

or

$$-\sinh\pi\bar{\mu}\,[\cosh\pi\bar{\mu}_{\mathrm{B}}H]^{-1}=Z_0\,\frac{\hbar\omega_0\cosh\pi\mu_{\mathrm{B}}H}{2kT\sinh(\hbar\omega_0/2kT)}\,\exp\left(\pi\bar{\mu}-\frac{D_1}{kT}\right). \tag{18}$$

In this equation we have taken into account the fact that $\mu < 0$, which follows from Eq. (17). Consequently $\mu - D_1 < 0$ and therefore in the conductivity band there is no degeneracy. Because of this, one may use Eq. (16) for χ_2.

It has been shown[4] that the approximation (4) is satisfied in all cases for values of H, such that $\mu_{\mathrm{B}}H < \Delta\epsilon$. Since we are considering very low temperatures $T \lesssim 10°$ K, for which $\Delta\epsilon > kT$, and large values of H, for which $\Delta\epsilon/\mu_{\mathrm{B}} > H$, and $\hbar\omega_0/2 > kT$ (usually for a semiconductor of the Ge type $m^* < m$, and $\hbar\omega_0/2 > \mu_{\mathrm{B}}H$) therefore, we can put $N/n_0 \ll 1$; $n_0 \gtrsim 10^{15}-10^{16}$. Smaller values of n_0 in the presence of an impurity band are hardly worth considering. But then from Eq. (18)

$$\exp\pi\bar{\mu}\approx 1-S\cosh\pi\bar{\mu}_{\mathrm{B}}H,\quad S=\frac{1}{n_0}\left\{\frac{\hbar\omega_0}{2kT}\cosh\pi\bar{\mu}_{\mathrm{B}}H\left(\sinh\frac{\hbar\omega_0}{2kT}\right)^{-1}\right\}e^{-D_1/kT}\ll 1, \tag{19}$$

i.e.,

$$\bar{\mu} \approx -(S/\pi)\cosh \pi\mu_{\mathrm{B}}H, \quad |\mu| \ll 1, \; N \approx n_0 S(1 - S\cosh \pi\bar{\mu}_{\mathrm{B}}H).$$

Consequently, for small values of b, $\bar{\mu}$ is negative and has a small absolute value, which decreases with increasing H/T. As for the susceptibility χ_2, for large values of H, when $N \ll n_0$ and consequently Eq. (19) holds, one finds

$$\chi_2(H, T) \approx \frac{N}{\pi\bar{\mu}_{\mathrm{B}}H}\left[1 + \pi\bar{\mu}_{\mathrm{B}}H \tanh \pi\bar{\mu}_{\mathrm{B}}H - \frac{\hbar\omega_0}{2} \coth \frac{\hbar\omega_0}{2kT}\right]\frac{\mu_{\mathrm{B}}}{H},$$

i.e., $|\chi_2| \ll n_0\mu_{\mathrm{B}}/H \equiv \chi_{\mathrm{sp}}$. Thus, in deriving an expression for χ_2, one may neglect corrections to $\bar{\mu}$ of the order of b_0^2. On the basis of what has been established thus far, taking into account the monotonic decrease of $|\bar{\mu}|$ with increasing H, one can readily determine the behavior of the magnitude of χ in a changing field H. The magnetic susceptibility χ_1, consists of three parts. The first term of Eq. (15) can be interpreted basically as the orbital susceptibility of the electrons in the impurity band, χ_0. As is evident from Eq. (15), with increasing H, χ_0 decreases in magnitude and oscillates like $\Phi'(\alpha)/\alpha$ for $\alpha \gtrsim 1$, i.e., for $H \gtrsim H_0$. Incidentally, since $n_0 \sim 10^{15} - 10^{18}$ and $a \sim 10^{-5} - 10^{-6}$, it follows that $H_0 = \hbar c/ea^2$ is not very large, i.e., $H_0 \sim 10^3 - 10^5$. The second term in Eq. (15) which arises from the magnetic susceptibility of the spin of the electrons in the impurity band, monotonically decreases with increasing H. The third term $\chi_{0\mathrm{sp}}$ arises as a consequence of the coupling between the orbital and spin magnetism. For large H the separation of χ into an orbital and spin susceptibility is ambiguous. This third term also oscillates with increasing H and its magnitude decreases, provided that $H \gtrsim H_0$. For sufficiently large H, but for $D_1 \gg \mu_{\mathrm{B}}H$, χ_2 plays a smaller role than χ_1. Since χ_0 and χ_{sp} are of the same order of magnitude, it can be shown, that the complete expression for χ decreases and changes sign, i.e., it oscillates.

Note that if the impurity has an average valence ξ, such that $1 \leq \xi \leq 2$ (for T = 0 in a narrow band ξ take the place of n_0), for calculating N(H, T) and χ_2 one ought to use the Fermi distribution function for the electrons in the conduction band. This is readily done, using the methods of contour integration.[6] However, by means of such calculations one can verify that the basic conclusion concerning the oscillatory behavior of $\chi(H)$ is still valid. In fact, in case that $1 \leq \xi \leq 2$ the magnitude of $\bar{\mu}$ has an oscillatory dependence on H only in the approximation, in which χ is proportional to b_0^4 and $|\chi_2| \ll |\chi_1|$, for sufficiently large D, large $\hbar\omega_0$ and low temperatures.*

Note also, that since H_0 is not exceedingly large, then for sufficiently low T, it is also possible that $H \gtrsim H_0$, but $kT \gg \mu_{\mathrm{B}}H$ and $kT \gg \hbar\omega_0$.

It should be emphasized that the principal result, i.e., the oscillation of Δn and $\chi(H)$ with increasing H, is obtained for $H_0 \lesssim H \ll H_1 \equiv \Delta\epsilon/\mu_{\mathrm{B}}$. This is the basic criterion which leads to the oscillation of χ. Since $\Delta\epsilon$ and H_0 can be practically independent of T, so also the oscillation of χ, generally speaking, can be realized for $\mu_{\mathrm{B}}H \gtrsim kT$ as well as for $\mu_{\mathrm{B}}H \lesssim kT$, so long as $H_0 \lesssim H \ll H_1$.

For example, if $\mu_{\mathrm{B}}H_0 < kT$, it is possible to have $\mu_{\mathrm{B}}H > kT$, as well as $\mu_{\mathrm{B}}H_0 \lesssim \mu_{\mathrm{B}}H < kT$, provided that, $\mu_{\mathrm{B}}H_1 > kT$. If on the other hand $\mu_{\mathrm{B}}H_0 > kT$, then it must be also that $\mu_{\mathrm{B}}H > kT$: for such low temperatures usually $\mu_{\mathrm{B}}H_1 = \Delta\epsilon > kT$, where $\Delta\epsilon$ is the energy gap between the fundamental impurity band and the conduction band[5] (it is possible to convince oneself, that for normal conditions, in the presence of a fundamental impurity band, rather than impurity levels, excitation of the impurity band does not play a very important role).

It is equally possible for $\hbar\omega_0 > kT$ or $\hbar\omega_0 < kT$, while still being consistent with the condition $H_0 \lesssim H \ll H_1$. Note, however, that for elementary semiconductors like Ge $m^* < m$ and $\hbar\omega_0/2 > \mu_{\mathrm{B}}H$. Since the change in the resistivity in a magnetic field, $\Delta\rho = \rho_H - \rho_0$ is proportional to Δn, and since Δn for $H_0 \lesssim H \ll H_1$ oscillates with increasing H, one can expect that in this case $\Delta\rho$ will also oscillate with increasing H (if $H_0 \lesssim H \ll H_1$). Therefore, if $\mu_{\mathrm{B}}H \ll kT$, then

$$\Delta\rho \sim \Delta n \approx \frac{b^4}{6\pi}\int_0^\infty d\beta \frac{\beta^4}{\sinh\beta} \sin \bar{\mu}_0\beta\Phi(\alpha).$$

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*Note that for $H \ll H_0$ and $\mu_{\mathrm{B}}H \ll kT$, $\chi_0 = S_0(1 - \tanh^2 \pi\bar{\mu}/2)(1 - 2\tanh^2 \pi\bar{\mu}/2)$, the magnitude of χ_0 can be[8] positive for $\bar{\mu} = 0$ (i.e., an approximately half-filled band), and it can be negative for $\bar{\mu} \to \pm\infty$ (i.e., for an almost empty, or an almost filled band: $n/n_0 \ll 1$ or $n/n_0 \approx 2$).

Thus the oscillation of χ with increasing H occurs only if $H_0 \lesssim H \ll H_1$, where H_1 is a particular value of the magnetic field, whose value is limited by the requirement that it should satisfy the approximation in Eq. (4), i.e., for $H_0 \ll H_1$.

The oscillations of $\chi(H)$ are completely distinguishable from the de Haas-van Alphen oscillations for the following reasons: (1) they do not depend on the degree of degeneracy of the electron gas, (2) the period $2H_0$ of these oscillations does not depend on H or T and their amplitude decreases with increasing T, or, more exactly, with decreasing b_0, and (3) from a calculation of the susceptibility for the special case (1) one can prove that the oscillations, by nature, occur as a consequence of the periodic dependence of $\epsilon(\mathbf{k})$ and of the non-vanishing character of the commutator $[\hat{A}, \hat{B}]$. The latter provides the basis of the assertion that in the case of narrow energy bands for any periodic dependence of $\epsilon(\mathbf{k})$, the susceptibility $\chi(H)$ will have an oscillatory part if $H_0 < H < H_1$. In effect, for any periodic behavior of $\epsilon(\mathbf{k})$, the approximate Hamiltonian for $H < H_1$, can be written as

$$\mathscr{H} = \sum_i A_i \cos\left(\mathbf{k}\mathbf{a}_i + \frac{e}{2\hbar c}[\mathbf{H}, \mathbf{r}]\,\mathbf{a}_i\right) + s\mu_{\mathrm{B}}H,$$

Here $\mathbf{r} = i\nabla_{\mathbf{k}}$ (The summation occurs for the nearest neighbor approximation, for example, in the model of tightly bound electrons; it is sufficiently good for an investigation of narrow bands).

It can be shown that

$$[(\mathbf{k}\mathbf{a}_i), ([\mathbf{H}\mathbf{r}]\,\mathbf{a}_i)] = 0;$$

where it is understood, that

$$\left[\cos\left(\mathbf{k}\mathbf{a}_i + \frac{e}{2\hbar c}\,[\mathbf{H}\mathbf{r}]\mathbf{a}_i\right),\ \cos\left(\mathbf{k}\mathbf{a}_j + \frac{e}{2\hbar c}\,[\mathbf{H}\mathbf{r}]\,\mathbf{a}_j\right)\right] \neq 0$$

for $i \neq j$. But, in agreement with item (3) above, the oscillatory terms in $\chi(H)$ are obtained by an evaluation of such commutators. Therefore in the case of a narrow band, i.e., for $|A_i/kT| < 1$, consider the case in which Z_H and χ can be evaluated by expanding them in powers of $|A_i/kT|$, and their solution can be obtained using the method of Feynman.[5] In this case, one can expect that $\bar{\mu}$ depends on H, just as before [see Eqs. (17) and (19)], and that the susceptibility χ has an oscillatory part, so that these oscillations give rise to the characteristics referred to above [items (1) and (2)]. To this same degree of precision, Eq. (18) still determines $\bar{\mu}$, even for the general case.

It must be understood, that these assertions require a more detailed investigation.

Let us return now to an examination of the oscillation of $\chi(H)$ in a semiconductor with an impurity band.

It follows from Refs. 9 and 10 that for moderate concentrations of impurities, n_0, when it is still possible to distinguish the impurity band from the conduction band, the random distribution of the impurities leads to a smearing of the edges of the impurity band. This leaves the level-distribution density $g(\epsilon)$ in most of the band nearly equal to the level-distribution density for an ordered distribution of the impurities. The impurities in semiconductors of the Ge-type are of the lattice-substitution type. Thus, while $g(\epsilon)$ has a substantial maximum near the middle of the impurity band, it becomes very small in the smeared edges. If kT is smaller than the width of the principal part of the smeared band, which forms the boundary of the impurity band, it is indeed dangerous to make use of the periodic approximation for $\epsilon(\mathbf{k})$. If, on the other hand, kT is greater than the width of the whole band, which is true for a narrow band, then the smearing of the band probably plays a small role, as a consequence of the small value of $g(\epsilon)$ in the smeared edges of the band.

To calculate the summation of states for $b_0 \ll 1$, it is therefore possibly advisable, to make use of the quantity $g(\epsilon)$, calculated for a particular periodic function, $\epsilon(\mathbf{k})$, which approximates the dispersion law in the impurity band. It is here that it seems to make sense to use the periodicity of $\epsilon(\mathbf{k})$ in the impurity band.

Conversely, if under suitable conditions (low T, $H_0 < H < H_1$, large D and small m^*) the predicted oscillations of $\chi(H)$ are observed experimentally in a semiconductor with an impurity band, this will probably allow one to conclude that $\epsilon(\mathbf{k})$ is nearly periodic. It is possible that oscillations, like those described here, will also occur in other solids, whose energy spectrum contains narrow energy bands. For this to occur, it must also be assumed that the parameter similar to a should have a value, such that $H_0 \ll H_1$. The effect will then occur if $H_0 < H < H_1$. One can anticipate that for semiconductors with impurity bands similar oscillations will also occur in some of the galvanomagnetic phenomena.

In conclusion, I wish to express my heartfelt thanks to Professor A. G. Samoilovich for his constant interest and valuable discussions.

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ELECTRON IN A PERIODIC ELECTRIC AND HOMOGENEOUS MAGNETIC FIELD, II.

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The energy spectrum of an electron in a magnetic field is considered in the case in which its trajectory in **k**-space (i.e., the intersection of the Fermi surface by a plane perpendicular to the magnetic field) is an open periodic. In this case, equidistant or quasi-equidistant discontinuities appear in the continuous spectrum. The distance between them is of the same order as that between levels in a discrete spectrum [$\sim \epsilon^2$, Eq. (3)], while the width, which is usually exponentially small or of order ϵ^4, may be of the order of ϵ^3 or ϵ^2 in individual cases. The results that are obtained are applied to the theory of the de Haas-van Alphen effect.

THE present paper is a continuation of the research of Ref. 1, in which the energy levels of an electron in a crystal in a magnetic field were found in the quasi-classical approximation for the case in which the trajectory in **k**-space is closed and is located entirely within a single cell of the reciprocal lattice. In this case the energy is determined from the equations obtained in the work of Lifshitz and Kosevich:[2]

$$S(E, k_3) = (2\pi eH/\hbar c)(n+\gamma) \equiv (2\pi/\alpha_0^2)(n+\gamma). \qquad (1)$$

Here $S(E, k_3)$ is the area of the intersection of the surface of constant energy in **k**-space by a plane perpendicular to **H**, while γ, as shown in Ref. 1, is equal to ½ for almost all closed trajectories.

It was shown in Ref. 1 that if the closed surfaces in neighboring cells of **k**-space come close to one another, the discrete levels $E = E(n, k_3)$ [which are obtained from Eq. (1)] broaden out into narrow bands. This broadening is small if the trajectories are closed. If the trajectory of the electron is an open periodic, then the broadening is so great that one no longer speaks of broadening but rather of discontinuities in a continuous spectrum.

Let us consider open periodic trajectories obtained in the intersection of open, isoenergetic surfaces of the "corrugated cylinder" type by planes which pass through the axis of the cylinder. When **H** lies in one of the principal crystallographic planes, as is usually the case for experimental investigations of the de Haas-van Alphen effect, the plane perpendicular to **H** is located in precisely this fashion.

1. THE LIMITING CASE OF A "WEAKLY CORRUGATED CYLINDER"

If the surface $E(k_1, k_2, k_3) = \text{const}$ is a cylinder (whose axis we identify with the k_2 axis), then the intersection of such a surface with a plane passing through the axis of the cylinder (we shall consider this plane perpendicular to the k_3 direction) will consist of two parallel lines $k_1 = \pm k_0 = \text{const}$.

As was shown in Refs. 1 and 2, $\alpha_0^2 k_1$ plays the role of the momentum relative to the "coordinate" k_2. Constancy of the classical momentum denotes free motion of a particle with a continuous energy spectrum. Continuity of the spectrum in our case follows from the following considerations.

In the absence of a magnetic field, let the energy spectrum of the electron be expressed by the equation

$$E(k_1, k_2, k_3) = \sum_{\mathbf{n}} A_{\mathbf{n}} e^{i\mathbf{kn}}. \qquad (2)$$

Then, as was shown in Ref. 1, the energy levels E in a magnetic field, and the eigenfunctions of the electron in **k**-space [under the condition that $\epsilon \ll 1$, where

$$\varepsilon = a/\alpha_0, \quad \alpha_0 = \sqrt{\hbar c/eH} \qquad (3)$$

a = lattice constant; if $a = 2.5 \times 10^{-8}$ cm, then $\epsilon = 10^{-2}$ for $H = 10^4$ Oe] are determined by the equation*

$$\hat{E}g \equiv \mathrm{E}(\hat{\varkappa}_1, k_2, k_3)\, g = \sum_{\mathbf{n}} A_{\mathbf{n}} \exp\left(i\,\mathbf{kn} - i\,\frac{n_1 n_2}{2\alpha_0^2}\right) g\left(k_2 - \frac{n_1}{\alpha_0^2}\right) = Eg; \qquad (4)$$

$$\hat{\varkappa}_1 = k_1 - (i/\alpha_0^2)\,\partial/\partial k_2. \qquad (5)$$

If the surface $\mathrm{E}(\kappa_1, k_2, k_3) = \mathrm{const}$ is a cylinder (with axis k_2), then $A_{n_1 n_2 n_3} = 0$ $(n_2 \neq 0)$ in Eq. (2), and Eq. (4) is satisfied by a plane wave, which it is appropriate to write in the form

$$g(k_1, k_2, k_3) = \exp\{i\alpha_0^2 (p_1 + k_1) k_2\},$$

so that

$$E = \sum_{\mathbf{n}} A_{\mathbf{n}}\, e^{i(p_1 n_1 + k_3 n_3)},$$

i.e., a continuous spectrum exists.

If a weak periodicity ("corrugation," Fig. 1) appears on the energy surface, then the classical momentum κ_1 becomes a periodic function of k_2. For a quadratic dependence of E on κ_1, this would mean motion in a weak periodic potential field. The eigenfunction is obtained in this case by multiplication of a plane wave by a modulating periodic function.

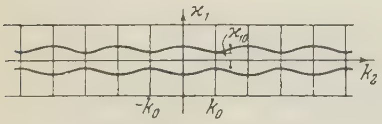

FIG. 1

Narrow discontinuities appear in the energy spectrum, and the continuous spectrum breaks up into bands which represent strongly broadened levels.

Considering terms with $n_2 \neq 0$ in Eq. (4) as a small perturbation, we can find the energy spectrum both close to the discontinuity and far from it. The discontinuities are obtained for those values of k_1 which satisfy the condition $k_1 = a_2 n/2\alpha_0^2$ (n = interger, a_2 is the lattice constant in the k_2 direction). This means that the areas of the intersections (in a single cell) appropriate to the middle of the level, are equal to $S = 2k_1(2\pi/a_2) = (2\pi/\alpha_0^2)(n + 1/2)$, i.e., they are again determined by Eq. (1).

The width of the n-th discontinuity in the spectrum is determined by the equation

$$E(k_1) = E^0(k_1) \pm \Big[\sum_{n_1 n_2 m_1 m_2} A_{n_1 n n_3} A_{m_1 n n_3} \exp\Big\{i\,a_2 n\,\frac{m_1}{\alpha_0^2} + ik_3(n_3 + m_3)\Big\}\Big]^{1/2}, \qquad (6)$$

where $k_1 = a_2 n/2\alpha_0^2$. The width of the discontinuity can be changed both monotonically and non-monotonically for a change in the number n or the magnetic field (α_0).

In the approximation considered, the discontinuities in the energy spectrum are quasi-equidistant, and have the order of magnitude ϵ^2. Such large discontinuities can take place only in those cases in which the corresponding coefficient A_{n_2} is different from zero. However, even if there is a finite number of terms in the expansion (2), exponentially small discontinuities exist in every case.

As was shown in Ref. 1 (see Mathematical Appendix 3),† Eq. (4) together with the nondiagonal terms corresponding to other energy bands, is written in the form

$$\hat{E}_r g_r - E g_r = \alpha_0^{-2} \sum_s \sum_{\mathbf{m}} D_{\mathbf{m}rs}\, e^{i\mathbf{km}}\, g_s\left(k_2 - \frac{m_1}{\alpha_0^2}\right).$$

The nondiagonal terms give discontinuities in the spectrum of order ϵ^3, in which the number of such discontinuities is equal to the number of coefficients $D_{m_1 m_2 m_3}$ not equal to zero $(\mathbf{m} \neq 0)$ (the coefficients $D_{\mathbf{m}}$ are the coefficients of the Fourier expansion $\Sigma D_{\mathbf{m}} e^{i\mathbf{km}}$ of the integral of some bilinear combination of eigenfunctions in the absence of a magnetic field: $\psi_{\mathbf{k}s}$, $\psi_{\mathbf{k}r}$).

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*This equation was obtained in Ref. 1 from the Schrödinger equation for an electron in a periodic electric and homogeneous magnetic field. For lack of space, we have not carried through this derivation, which demonstrates that we can obtain this Hamiltonian if we postulate the substitution in Eq. (2) of the corresponding non-commuting operators for the components of k, carrying out the operation of total symmetrization, i.e., arbitrary products of $\hat{k}_i^n$ and $\hat{k}_s^m$ are taken in all possible combinations.

†We take the opportunity to observe that there was an error [in Eq. (8) of Appendix 2] in that paper which has no effect on the subsequent conclusions.

In all the cases considered in the present section, the discontinuities in the spectrum arise for those alues of the energy for which the area of the intersection of the isoenergetic surface is given by $S = \pi n/\alpha_0^2$, and the mean of the broadened level corresponds to an area of intersection $S = (2\pi/\alpha_0^2)(n + \frac{1}{2})$, recisely the same as for the discrete levels of the closed trajectory.

In the following sections we shall consider cases in which the discontinuities in the spectrum are suf-iciently large that a noticeable de Haas-van Alphen effect appears. In such a case, we shall not make use f the approximation of a "weakly corrugated cylinder," since it requires too large an anisotropy of the co-fficients, expressed by the relation

$$A_{n_1n_2n_3} \ll \varepsilon^2 A_{n_10n_3}. \tag{7}$$

uch an anisotropy obviously does not occur in real cases ($\epsilon^2 \lesssim 10^{-4}$).

2. SOLUTION OF THE PROBLEM OF MOTION IN A PERIODIC FIELD, USEFUL FOR SMALL VALUES OF THE MOMENTUM ON THE CELL BOUNDARIES

In the case of closed trajectories which are divided into large regions of imaginary momentum, the roadening associated with a periodicity in the direction k_2 (Fig. 2) is exponentially small (in this case, owever, as was shown in Refs. 1 and 5, there is an additional broadening, of order ϵ^4 and connected with the non-equivalence in the crystal of the different positions of the "center of rotation" of the electron). Such a picture holds for the motion of a particle in a periodic potential field at low energy levels, when the regions of imaginary momentum are large (level A in Fig. 2). The bands in this case are so narrow that they can be regarded as narrow levels. When the closed trajectories are divided into small regions (level B in Fig. 2), the broadening becomes significant. In this case, we must use a method analogous to the method of "strongly coupled lectrons" for the calculation of the broadening.

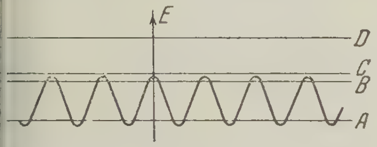

FIG. 2

If the trajectories are open and the momentum is large (which corresponds to the level D), the dis-ontinuities are small.

We now consider the case in which the momentum in the vicinity of some point (we shall assume that it s located on the cell boundary) takes on a small value (level C).

We analyze several neighboring elementary cells in the region shown in Fig. 3. While the momentum κ_1 is large in regions I, III, V, and the quasi-classical approximation can be used, the momentum is small in regions II and IV, and we can obtain an exact solution. A similar division into different regions follows in the case in which κ_1 or κ_1' has a discontinuity for some $k_2 = b$ (Fig. 3). Our considerations will apply to both cases.

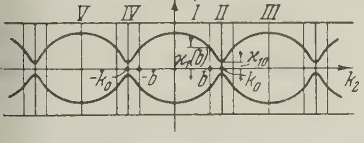

FIG. 3

We write the eigenfunctions for regions I, III, V in the form of linear combinations of quasi-classical wave functions.

The second quasi-classical approximation for these functions, obained in Ref. 1, has the form:

$$g(k_1k_2k_3) = e^{i\alpha_0^2 k_1k_2} f(k_2), \quad f(k_2) = \left|\frac{\partial E}{\partial \varkappa_1}\right|^{-1/2} \exp\left\{-i\alpha_0^2 \int_0^{k_2} \varkappa_1\, dk_2\right\}, \tag{8}$$

while the next (third) approximation, which we employ in the present paper, is

$$f(k_2) = \left|\frac{\partial E}{\partial \varkappa_1}\right|^{1/2} \exp\left\{-i\alpha_0^2 \int_0^{k_2} \varkappa_1\, dk_2\right\} \left(1 - \frac{i}{2\alpha_0^2} \int_0^{k_2} F(k_2)\, dk_2\right), \tag{9}$$

where

$$F = Q\left(\frac{3}{4}\frac{P'^2}{P^3} - \frac{P''}{2P^2}\right) - \frac{Q'P'}{2P^2} + \frac{Q''}{4P} + \frac{R\varkappa_1''}{12P}; \quad P = \frac{\partial E}{\partial \varkappa_1}, \quad Q = \frac{\partial^2 E}{\partial \varkappa_1^2}, \quad R = \frac{\partial^3 E}{\partial \varkappa_1^3}.$$

We write the eigenfunction in regions II, IV in the form $f = Af_1 + Bf_2$, where f_1 and f_2 are the even and odd solutions, respectively, in these regions. From the condition of equality of the eigenfunctions and their derivatives on the boundaries of the regions, we get the equation (p is a new, continously variable quantum number corresponding to a quasimomentum in a periodic field):

$$\cos p = \left\{f_1' f_2 + f_1 f_2' + f_1 f_2 \frac{P'}{P} - \frac{\Phi}{\alpha_0^4}\left[\frac{1}{2}(f_1 f_2)' \frac{P'}{P\varkappa_1} + \frac{f_1' f_2'}{\varkappa_1} + f_1 f_2\left(\frac{P'^2}{4P^2\varkappa_1} - \alpha_0^4 \varkappa_1\right)\right]\right\} \frac{1}{D} \cos\frac{\alpha_0^2 S(b)}{2}$$
$$- \left\{\frac{P'}{2P\varkappa_1}(f_1 f_2)' + \frac{f_1' f_2'}{\varkappa_1} + f_1 f_2\left(\frac{P'^2}{4\varkappa_1 P^2} - \alpha_0^4 \varkappa_1\right) + \Phi\left[(f_1 f_2)' + f_1 f_2 \frac{P'}{P}\right]\right\} \frac{1}{\alpha_0^2 D} \sin\frac{\alpha_0^2 S(b)}{2}. \qquad (10)$$

Here

$$S(b) = 2\int_{-b}^{b} \varkappa_1 dk_2, \;\; D = f_1 f_2' - f_1' f_2, \;\; \Phi = \int_0^b F dk_2, \;\; P = \frac{\partial E}{\partial \varkappa_1}, \qquad (11)$$

and the values of all quantities are referred to the point $k_2 = b$.

This equation defines the possible values of the areas of intersection, i.e., it gives the possible values of the energy in implicit form. We write Eq. (10) for the case in which $\kappa_1 = \kappa_{10}$ is small on the cell boundary (for $k_2 = k_0$).

We expand $E(\kappa_1, k_2, k_3)$ in a series about the point (κ_{10}, k_0) (retaining only the terms we have written down):

$$E(\varkappa_1, k_2, k_3) = E(\varkappa_{10}, k_0, k_3) + (\varkappa_1 - \varkappa_{10})\frac{\partial E}{\partial \varkappa_{10}} + \frac{1}{2}(\varkappa_1 - \varkappa_{10})^2 \frac{\partial^2 E}{\partial \varkappa_{10}^2} + \frac{1}{2}(k_2 - k_0)^2 \frac{\partial^2 E}{\partial k_0^2} + \cdots \qquad (12)$$

Here the terms of first degree in $k_2 - k_0$ are absent in view of the obvious [following from Eq. (2)] vanishing of the corresponding coefficients: $\partial E/\partial k_0$ and $\partial^2 E/\partial \kappa_1 \partial k_0$.

We shall consider that the expansion (12) is sufficiently accurate up to the point $k_2 = b$; in the opposite case, we can keep the succeeding terms of the expansion (which would naturally make the subsequent calculations more involved). Inasmuch as the expansion (12) refers to points on a curve of constant energy, it determines the dependence $\kappa_1(k_2)$. In our case, this dependence has a hyperbolic form, the equation of which is written:

$$\varkappa^2 = \varkappa_0^2 + (\varkappa_0/R)\, a_2^2 (k_2 - k_0)^2. \qquad (13)$$

Here, $\kappa = a_1\kappa_1$, $\kappa_0 = a_1\kappa_{10}$ is the value of κ on the boundary of the cell (Fig. 3). R is the dimensionless radius of curvature of the curve $\kappa_1(k_2)$ at the point k_0. Replacing κ by the operator (5), we get a second-order differential equation:

$$\partial^2 f/\partial x^2 + (\lambda + x^2) f = 0. \qquad (14)$$

Here

$$x = a_2(k_2 - k_0)/\varepsilon (R/\varkappa_0)^{1/4}, \;\; \lambda = (\varkappa_0/\varepsilon)^2 \sqrt{R/\varkappa_0}. \qquad (15)$$

As is evident from (10), we are interested in the values of the function f and its derivatives at the point $k_2 = b$, for which $\kappa \gg \kappa_0$ and, consequently, in accordance with Eqs. (13) and (15),

$$\varkappa^2 \approx (\varkappa_0/R)\, a_2^2 (b - k_0)^2 = \varepsilon^2 x^2 \sqrt{\varkappa_0/R}.$$

Making the substitution

$$z = -ix^2, \;\; f = e^{-z/2} u(z), \qquad (16)$$

in Eq. (14), we get an equation for $u(z)$:

$$zu'' + u'(1/2 - z) - \frac{\lambda + i}{4i} u = 0, \qquad (17)$$

from which it follows that u is a degenerate hypergeometic function.[4] This allows us to write the exact solution of Eq. (14). The two independent solutions, the even f_1 and the odd f_2, have the form

$$f_1 = e^{-z/2} F(\alpha, \gamma, z), \;\; f_2 = e^{-z/2} z^{1/2} F(\alpha - \gamma + 1, 2 - \gamma, z), \qquad \alpha = (\lambda + i)/4i, \;\; \gamma = 1/2. \qquad (18)$$

Here

$$F(\alpha, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} (-z)^{-\alpha} G(\alpha, \alpha - \gamma + 1, z) + \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^{z} z^{\alpha - \gamma} G(\gamma - \alpha, 1 - \alpha, -z),$$

where

$$G(\alpha, \beta, z) = \frac{\Gamma(1 - \beta)}{2\pi i} \int\limits_{C} \left(1 - \frac{t}{z}\right)^{-\alpha} t^{\beta - 1} e^{t}\, dt, \tag{19}$$

and the integration is carried out over the contour C surrounding the point $t = 0$ and extending on both branches along the real axis to $-\infty$. The function $G(\alpha, \beta, z)$ can be represented in the case of large z by an asymptotic series:

$$G(\alpha, \beta, z) = 1 + \frac{\alpha\beta}{z} + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{z^2} + \dots \tag{20}$$

We assume that the expansion (12) [and that also means Eq. (14)] remains valid up to such large values of $\kappa_1(b)$ (Fig. 3) for which Eq. (9) for the quasi-classical solution is also valid. The corresponding approximation for the exact solution will be such for which we keep only the first and second terms. The functions (18) in this approximation take on the form

$$f_1 = \operatorname{Re}(BF), \quad f_2 = \operatorname{Re}(CF), \quad B = \Gamma\left(\frac{1}{2}\right) \Big/ \Gamma\left(\frac{1}{4} + i\frac{\lambda}{4}\right), \quad C = e^{-i\pi/4}\, \Gamma\left(\frac{3}{2}\right) \Big/ \Gamma\left(\frac{3}{4} + i\frac{\lambda}{4}\right),$$

$$F = \exp\left\{-\frac{1}{2}\ln x - \frac{\pi i}{8} + i\frac{\lambda}{2}\ln x + i\frac{x^2}{2}\right\}\left(1 - \frac{3i + 4\lambda - i\lambda^2}{16x^2}\right) \tag{21}$$

Computing the brackets in Eq. (10) [by means of (21)] and making use of the properties of the Γ functions, we transform (10) to the form

$$\cos p = \sqrt{1 + e^{-\pi\lambda}} \left\{\cos\frac{\alpha_0^2}{2} S + \frac{\lambda^2 - 6}{8x^2} \sin\frac{\alpha_0^2}{2} S\right\}. \tag{22}$$

Here S is the area of intersection (in one cell) of the surface $E = \text{const}$ with the plane $k_3 = \text{const}$, $x^2 = (\kappa/\epsilon)^2\sqrt{R/\kappa_0}$, λ is from (15), κ_0 is the dimensionless momentum on the boundary of the cell, κ, that on the boundary of regions I — II (Fig. 3). Expansion (20), and therefore (21) also, is valid for $\lambda \ll x$.

The boundary of region II is that point $k_2 = b$ where the dispersion laws (12) or (13) are still valid. It is assumed that at this point, the quasi-classical approximation (9) is also valid ($\kappa \sim \epsilon^{1/2}$ usually satisfies both these conditions). If the quadratic expansion (12) takes place even for κ larger than $\sim \epsilon^{1/2}$, then we must take the point with the largest possible κ as the boundary for the region, since the larger the κ, the more accurate both the asymptotic expansion (20) and the quasi-classical solution (9) will be.

Initially, we throw away the small terms ($\sim x^{-2}$) in (22). For $\lambda \ll 1$, we then obtain $\cos p = \sqrt{2} \times \cos(\alpha_0^2 S/2)$. Thus, for very low levels, the allowed and forbidden intervals are identical, with accuracy up to the discarded terms. Upon increase in λ, the width of the forbidden intervals decreases; for $\lambda \gg 1$, the forbidden intervals are already determined by terms $\sim x^{-2}$ and also, generally speaking, decrease with increase in energy.

The smaller the ratio R/κ_0, the larger the number of levels located in regions of those values of λ for which the square root $\sqrt{1 - e^{-\pi\lambda}}$, is still significantly different from unity.

If we keep terms $\sim x^{-4}$ in (22), then, as is shown, this reduces not only to the addition of similar terms to the coefficient of the sine in (22), but also to the appearance in (22) of a new component of the form $x^{-4} e^{-\pi\lambda/2} \sin(\alpha_0^2 S(b)/2)$ [$S(b)$ is the area within the boundaries $(-b, +b)$]. The chief peculiarity here is the fact that the frequency with which it oscillates (for a change in the energy or magnetic field), differs little from the frequency of the basic terms of (22). This leads to a non-monotonic change in the width of the forbidden intervals for a change in the magnetic field. A similar phenomenon has already been pointed out in Sec. 1.

3. CASE IN WHICH THE MOMENTUM OR ITS DERIVATIVE HAVE DISCONTINUITIES

Let the momentum κ_1 or its derivative $\kappa_1' = d\kappa_1/dk_2$ have a discontinuity at $k_2 = b$ (three individual cases are drawn in Fig. 4). For the functions f_1 and f_2, we use the quasi-classical expression (8), which is valid in those cases when the momentum κ nowhere takes on too small a value ($\kappa \gg \epsilon$).

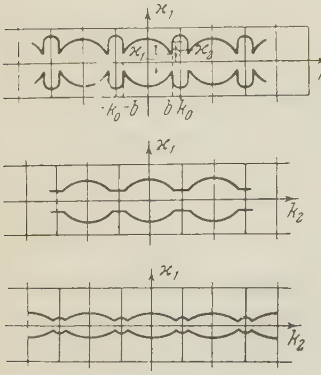

FIG. 4

We introduce the notation

$$y = \frac{\alpha_0^2}{2} S/2,\ y_1 = y - y_b,\quad y_b = \alpha_0^2 S(b)/2,\ P_1 = \partial E/\partial\varkappa_1,\quad P_2 = \partial E/\partial\varkappa_2, \tag{23}$$

κ_1 and κ_2 are the values of κ on the two sides of the discontinuity. We denote the derivaties with respect to k_2 by primes. With this notation, Eq. (10) takes on the form

$$\cos p = \cos y - \frac{(\varkappa_1 - \varkappa_2)^2}{2\varkappa_1\varkappa_2} \sin y_b \sin y_1 + \frac{P_2' P_1 - P_1' P_2}{2\alpha_0^2 P_1 P_2}\left(\frac{1}{\varkappa_1}\sin y_b \cos y_1 + \frac{1}{\varkappa_2}\cos y_b \sin y_1\right) + (P_1' P_2 - P_2' P_1)^2 \sin y_b \sin y_1 / 8\alpha_0^4 \varkappa_1 \varkappa_2 P_1^2 P_2^2. \tag{24}$$

If the momenta on the boundaries of the regions, κ_1 and κ_2, are small and we can apply there the quadratic dispersion laws $E = Q\kappa^2/2 + \varphi(k_2)$, then $P_1 = Q_1\kappa_1$, $P_2 = Q_2\kappa_2$ (Q_1 and Q_2 are constants) and the equation is simplified. We consider two particular cases:

1. $\kappa_1 \neq \kappa_2$, $\kappa_1' = \kappa_2'$. To abbreviate the expressions in this case from the general formula (24), we omit all terms except the first two (generally, the remaining terms are of much higher order of smallness; however, in special cases they can be large, and must then be retained). Returning to the earlier notation, we have:

$$\cos p = \cos\frac{\alpha_0^2}{2} S - \frac{(\varkappa_1 - \varkappa_2)^2}{2\varkappa_1\varkappa_2} \sin\frac{\alpha_0^2}{2} S(b) \cdot \sin\frac{\alpha_0^2}{2}[S - S(b)]. \tag{25}$$

2. $\kappa_1 = \kappa_2$, $\kappa_1' \neq \kappa_2'$. In this case, we get

$$\cos p = \cos\frac{\alpha_0^2}{2} S + \frac{\varkappa_2' - \varkappa_1'}{2\alpha_0^2 \varkappa_1^2} \sin\frac{\alpha_0^2}{2} S + \frac{(\varkappa_1' - \varkappa_2')^2}{8\alpha_0^4 \varkappa_1^4} \sin\frac{\alpha_0^2}{2} S(b) \cdot \sin\frac{\alpha_0^2}{2}(S - S(b)). \tag{26}$$

There are terms in Eqs. (25) and (26) which have a frequency different from the frequency of the fundamental term. As will be evident from what follows, their presence leads to fluctuations in the width of the levels upon change in the magnetic field or the energy.

Let us find what value one ought to expect for the derivative κ' at the discontinuity of κ and what value for the derivative κ'' one ought to expect for the discontinuity in κ'. The answer to this question furnishes the requirement (see Ref. 1) that in the expansions applied in the quasi-classical method (in powers of ϵ^2) each successive term would be much smaller than the preceding. Investigation shows that if at each point the inequality

$$\varkappa' \ll \varepsilon^{-2}\varkappa,\quad \varkappa'' \ll \varepsilon^{-4}\varkappa^4,\quad \varkappa'' \ll \varepsilon^{-2}\varkappa', \tag{27}$$

is not satisfied, then we must expect that a discontinuity occurs (either in κ or in κ') at this point.

4. APPLICATION TO THE THEORY OF THE DE HAAS-VAN ALPHEN EFFECT

In a previous paper,[5] starting from the approximations of electrons that are weakly and strongly coupled, it was shown that the broadening of the levels (if it does not change monotonically with change in energy and magnetic field H) leads to the phenomenon of "beats" on the curves of the dependence of the magnetic susceptibility and other physical quantities on H. In the approximation of weakly coupled electrons (in the anisotropic case) just such open trajectories occur about which we spoke in the preceding sections. In almost all cells, the curve $\kappa_1(k_2)$ is an ellipse and is sharply distorted only at the boundary.

In the present section the phenomenon of "beats" is again considered but outside the framework of the approximation of weakly or strongly coupled electrons. In Secs. 1 — 3 we considered a series of equations which determine the energy levels and contain terms of different frequency, which produce oscillation of the levels. We investigate the simplest of these – Eq. (25) [Eq. (26) is entirely similar].

Using the notation

$$\alpha_0^2S/2=y,\ \ (\alpha_0^2/2)(S-S(b)=\gamma y,\ \ (\varkappa_1-\varkappa_2)^2/2\varkappa_1\varkappa_2=2\beta, \tag{28}$$

we write (25) in the form:

$$\cos y-2\beta\sin\gamma y\cdot\sin y(1-\gamma)=\cos p. \tag{29}$$

Since γ and β do not depend on the magnetic field, it is possible to assume them constant and consider that only y changes. Let γ satisfy the inequality $\gamma \ll 1$. We consider two cases: $\beta \ll 1$ and $\beta \gg 1$. It is expedient to rewrite (29) in the form

$$\sqrt{1+4\beta(1+\beta)\sin^2\gamma y}\cos(y+\varphi_1)\equiv R\cos(y+\varphi_1)=\cos p;\ \sin\varphi_1=\frac{3\sin 2\gamma y}{R}. \tag{30}$$

For $\beta \ll 1$, the left side of (30), as well as the function $\varphi = y + \varphi_1$, has the form shown in Fig. 5.

It is not difficult to show from (30) that the area of intersection in Fig. 5, corresponding to the center of the level will be

$$S=\frac{2\pi}{\alpha_0^2}\left(n+\frac{1}{2}-\frac{\beta}{\pi}\sin 2\pi\gamma\left(n+\frac{1}{2}\right)\right). \tag{31}$$

The connection between S, n and p inside the level is expressed by the formula

$$n=\frac{\alpha_0^2S}{2\pi}+\frac{\beta}{\pi}\sin\gamma\alpha_0^2S-\frac{1}{\pi}\arccos(\cos p/R), \tag{32}$$

and the width of the level is equal to

$$\Delta y=\pi-4\sqrt{\beta\sin^2\pi n\gamma}. \tag{33}$$

The width of the level changes periodically with the change in the number of the level, n.

The effect of broadening on the de Haas-van Alphen effect can be studied by computing the partition function

$$Z(E,H)\sim\int dp\sum_{n=0}^{n\,\max}k_3=\sum_{q=-\infty}^{+\infty}\int dp\int dn e^{2\pi iqn}k_3(n,p).$$

We transform the inner integral to the form $\int dp\int dk_3\, e^{2\pi in(k_3,\,p)}$ and substitute for n from Eq. (32). We then obtain

$$-4\pi\beta|q|\cos[\alpha_0^2S_m+2\beta\sin(\alpha_0^2S_m\gamma)]\sin^2\left(\gamma\frac{\alpha_0^2S_m}{2}\right) \tag{34}$$

(S_m is the extremum area of intersection with the plane k_3 = const.)

Terms of such form all enter into the expression for the magnetic susceptibility instead of the simple $\cos(\alpha_0^2S_m+\delta)$ which is obtained in the absence of level broadening. It is evident that the broadening reduces to (1) a decrease in the amplitude of oscillation by a factor β, (2) the appearance of a factor $\sin^2(\gamma\alpha_0^2S_m/2)$ which leads to beats, and (3) the appearance of an additional small oscillating term under the sign of the cosine.

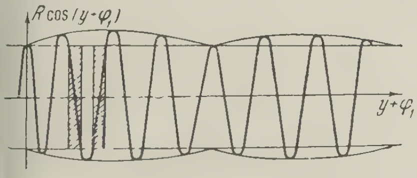

FIG. 5

The beats are not necessarily expressed by a factor similar to that used in (34) and capable of completely absorbing the oscillations (at absolute zero). This factor has another form in the case $\beta \gg 1$, when we get, in place of (34),

$$\cos\left[(1-\gamma)\alpha_0^2S_m-\frac{1}{A}\cos\frac{\alpha_0^2}{2}\gamma S_m\right]I_0(2/A),\quad A=\beta\sin\frac{\alpha_0^2}{2}\gamma S_m+\sin\frac{\alpha_0^2}{2}\gamma(1-\gamma)S\ . \tag{35}$$

Here I_0 is the Bessel function of order zero. Equation (35) is valid in the region in which the argument I_0 remains < 1. Therefore, I_0 does not vanish, but oscillates slightly, similar to what is obtained in the approximation of weakly coupled electrons (considered in Ref. 5). Thus we have made it clear that both complete and incomplete beats of oscillating physical quantities are possible (Fig. 6).

Both these forms of beats are encountered in experiment.[6,7] The beats can also be connected with the presence of two groups of electrons having neighboring parameters. If the beats are connected with the broadening of the levels, the period of oscillation of the curve $\chi(H^{-1})$ ought, generally speaking, to be much less than in the case in which these beats are produced by the existence of two electron groups.

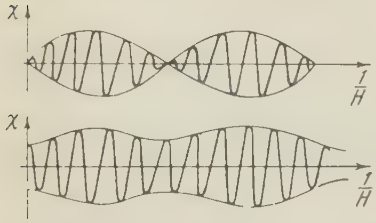

FIG. 6

Both beat mechanisms account equally well for the great sensitivity of the beats of the angle made by **H** with one of the crystalline axes (but the large sensitivity to the arrangement of the crystal, i.e., to the emergence of **H** from the fundamental crystallographic plane, is better accounted for by the mechanism described here). The vanishing of the beats on increase in temperature is also equally well explained. If we expand (34) into monochromatic components, then each of them (after integration with the Fermi distribution function in the calculation of the thermodynamic potential from the partition function) will have its own temperature factor. For sufficiently low temperatures, these factors are all close to unity, but upon elevation of the temperature, one of the factors remains significantly greater than the others and the beats disappear.

The dispersion law (34), can lead, for certain values of the parameters, to the "crescent-shaped" character of the change of the amplitude of oscillation of physical quantities observed in experiments.[6,7]

In this work we have not considered the interesting case in which the equation for $\kappa_1(k_2)$ is more than double-valued (for examples, quadruple valued). Such trajectories are frequently encountered.[8] In the dispersion equation for such trajectories, there appear terms with a large number of different frequencies. Corresponding to this case the oscillations of the physical quantities can have a more complicated character than is shown in Fig. 6.

The author takes this occasion to express his sincere gratitude to Professor I. M. Lifshitz for his attention to the present research and for his discussions.

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BEHAVIOR OF PARTICLES OF SMALL EFFECTIVE MASS IN SUPERFLUID HELIUM

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The kinetic equation for the interaction of electrons with thermal excitations in He II is solved by the Fokker-Planck method. The electric current density is determined for fields of various strengths.

THE study of the behavior of foreign particles in liquid He II is of considerable interest toward understanding the nature of their interaction with the elementary excitations. The only substance soluble in He II is He^3, the behavior of which in solution has already been investigated both experimentally[1] and theoretically.[2]

A. I. Shal'nikov has called attention to the possibility of studying the behavior in He II of light particles such as electrons, obtained through ionization of the helium by some type of source. If the effective mass m of the electron in solution is of the order of the value for the free electron, the average velocity of the electrons in a state of thermal equilibrium will be substantially greater than that of sound over virtually the entire temperature interval down to $T \sim 10^{-2}$. In such a case the interaction of the electrons with the phonons will take place primarily by a single-phonon mechanism, i.e., the electron can emit and absorb phonons, thereby altering its direction through some angle without appreciably changing its energy (we shall perform the corresponding calculations below). In its interaction with a roton the electron is scattered by the latter, again changing its energy only very slightly in the process.

This circumstance makes it possible to apply the Fokker-Planck method, in the form developed by Davydov,[3] to the solution of the kinetic equation.

It is possible to determine experimentally the mobility of charged particles in electric fields. Estimates show that for various reasonable fields and mean free paths the values to be expected for the electron drift velocity are much greater than the velocity of the normal component. In such a case the excitation gas may be regarded as at rest and in thermal equilibrium.

We shall seek a distribution function for the electrons in the form of a series in Legendre polynomials, limiting ourselves to the zero and first terms of the expansion (the convergence of the series is demonstrated in Ref. 3):

$$f = f_0 + f_1 \cos \vartheta \ldots +$$

Here ϑ is the angle made by the electron momentum with the direction of the electric field.

As Davydov has shown, the equations for f_0 and f_1 can be represented in the form

$$\frac{1}{3p^2} \frac{\partial}{\partial p} (p^2 eEf_1) = \frac{1}{p^2} \frac{\partial}{\partial p} p^2 \left(A_2 \frac{\partial f_0}{\partial p} + A_1 f_0 \right); \tag{1}$$

$$eE\, \partial f_0 / \partial p = -Bf_1; \tag{2}$$

$$B = \int (1 - \cos \psi)\, dw, \quad A_2 = \frac{1}{2} \int (\delta p)^2\, dw, \tag{3}$$

dw is the probability for scattering through an angle ψ, and δp is the corresponding change in the absolute value of the momentum.

The collision integral in Eq. (1) has, as it should, the form of a divergence in the space of absolute values of the momenta of a certain current

$$s = A_2 \partial f_0 / \partial p + A_1 f_0.$$

The coefficient A_1 may be found from the requirement that s fall identically to zero for the Maxwell distribution $f_0 \sim \exp(-p^2/2mkT)$, and turns out to be $A_1 = A_2 p/mkT$.

From (1) and (2) we obtain the equation for f_0:

$$\frac{1}{p^2}\frac{\partial}{\partial p}p^2\left[\left(\frac{e^2E^2}{3B}+A_2\right)\frac{\partial f_0}{\partial p}+A_2\frac{p}{mkT}f_0\right]=0.$$

The first integral corresponds to constant current s. The constant of integration is to be set equal to zero; in the alternative case the current goes to ∞ at p = 0.

From the equation obtained for f_0 the following quadrature is found:

$$f_0 = C\exp\left\{-\int_0^\infty \frac{A_2 p\, dp}{mkT\,(A_2+e^2E^2/3B)}\right\}. \tag{4}$$

The constant here is determined from the normalization condition. The function f_1 is readily expressed in terms of f_0 with the aid of (2) and (4):

$$f_1 = \frac{eE}{B}\,\frac{A_2 p}{mkT\,(e^2E^2/3B + A_2)}\,f_0.$$

Knowing f, it is simple to obtain the electric current density

$$j=\int e\frac{p\cos\vartheta}{m}fp^2dp\,do = \frac{4\pi}{3}\,\frac{e^2E}{m^2kT}\int_0^\infty \frac{f_0A_2p^4\,dp}{B\,(A_2+e^2E^2/3B)}, \tag{5}$$

from which it is possible to determine the electrical conductivity and the mobility of the particles $u = j/ne^2E$.

Equations (4) and (5) completely solve the problem, provided concrete expressions are known for A_2 and B; we now turn to the computation of these.

The probability for phonon emission is found from the usual perturbation theory formulas, where the interaction energy is to be set equal to

$$V = (\partial\Delta_e/\partial\rho)\tilde{\rho};$$

$\overline{\rho}$ is the variation in the density due to the presence of the phonon, and $\partial\Delta_e/\partial\rho$ corresponds to a dependence $\epsilon = \Delta_e + p^2/2m$ of the impurity (electron) energy upon the density.

Calculations similar to those performed by Zharkov and Khalatnikov[2] lead to the following formula for the probability of phonon emission:

$$dw_{\text{e ph}} = \frac{s}{\lambda}\,\frac{n+1}{8\pi m^3s^2}\,\delta(\mathbf{p}-\mathbf{p}'-\mathbf{q})\,\delta(\varepsilon-\varepsilon'-sq)\,d\mathbf{p}'\cdot q\,d\mathbf{q},\qquad n=(e^{-sq/kT}-1)^{-1},$$

n is the Planck distribution function for the phonons, which we take to be in thermal equilibrium, s is the velocity of sound, and the notation $\lambda = \pi\rho h^4/m^3(\rho\partial\Delta_e/\partial\rho)^2$ is introduced for the sake of brevity.

We integrate the probability expression over dq and over the absolute value of the momentum p′, obtaining as a result

$$dw_{\text{e ph}} = \frac{s}{\lambda}\,\frac{n+1}{2m^2s^2}\,qp'\,\frac{do}{4\pi}\Big/\left|\frac{\partial}{\partial\varepsilon'}(\varepsilon-\varepsilon'-sq)\right|. \tag{6}$$

The probability for absorption of a phonon is obtained from (6) by replacing n + 1 by n:

$$dw_{\text{a ph}} = \frac{s}{\lambda}\,\frac{n}{2m^2s^2}\,qp'\,\frac{do}{4\pi}\Big/\left|\frac{\partial}{\partial\varepsilon}(\varepsilon-\varepsilon'+sq)\right|. \tag{7}$$

From the energy and momentum conservation laws for the phonon emission process (these results are sufficiently accurate for the case of phonon absorption as well):

$$p^2/2m = p'^2/2m + sq,\quad \mathbf{p}-\mathbf{p}'=\mathbf{q}$$

we find

$$q^2 \simeq 2p^2(1-\cos\psi),\quad (\delta p)^2 \equiv (p'-p)^2 \simeq 2m^2s^2(1-\cos\psi), \tag{8}$$

where ψ is the angle between $\mathbf{p}$ and $\mathbf{p}'$.

Evidently $\delta p/p \sim ms/p \sim (ms^2/kT)^{1/2}$ which justifies the application of the Fokker-Planck method. We can therefore in Eqs. (6) and (7) set $p' \simeq p$ and the quantity

$$\left|\frac{\partial}{\partial\varepsilon'}(\varepsilon - \varepsilon' \pm sq)\right| \approx 1, \quad \text{since} \quad sq \sim sp \ll \frac{p^2}{2m}.$$

No difficulty is now encountered in determining $A_2^{\mathrm{ph}} = \frac{1}{2}\int(\delta p)^2(dw_{e\,\mathrm{ph}} + dw_{a\,\mathrm{ph}})$ with the aid of (8):

$$A_2^{\mathrm{ph}} = m^2s^2\int(1-\cos\psi)\frac{s}{\lambda}\frac{2n+1}{2m^2s^2}qp\frac{do}{4\pi} = \frac{s}{8\lambda p^3}\left(\frac{kT}{s}\right)^5\int_0^{2sp/kT} x^4 \coth\frac{x}{2}\,dx, \qquad x \equiv sq/kT. \tag{9}$$

In accordance with (8)

$$B^{\mathrm{ph}} = A_2^{\mathrm{ph}}/m^2s^2. \tag{10}$$

In considering the scattering of an electron by a roton it is reasonable to take their energy of interaction to be a δ-function of the distance between them. This leads, obviously, to isotropic scattering, the cross-section for which is independent of the electron and roton velocities; there is, moreover, no necessity for introducing a relative velocity for the electron and roton, since the velocity of the latter is always very small. The probability for scattering of the electron through some arbitrary angle can thus be represented in the form

$$dw = (v/l)\,do/4\pi, \quad 1/l = \sigma N_{\mathrm{r}}, \tag{11}$$

σ is some unknown cross-section, and N_{r} is the number of rotons per unit volume.

From this we obtain directly

$$B^{\mathrm{r}} = \int(1-\cos\psi)(v/l)\,do/4\pi = v/l.$$

To compute A_2^{r} it is necessary to find the value of $\langle(\delta p)^2\rangle_{\mathrm{r}}$, averaged over the roton distribution. The energy conservation law yields (with $\mathbf{P}$ the roton momentum):

$$-p\delta p/m = [P'^2 - P^2 - 2P_0(P'-P)]/2\mu. \tag{12}$$

$\mathbf{P}'$ can be found from the momentum conservation law

$$\mathbf{P}' = \mathbf{P} - \Delta\mathbf{p}; \quad \Delta\mathbf{p} \equiv \mathbf{p} - \mathbf{p}'; \quad \Delta\mathbf{p}^2 \approx 2p^2(1-\cos\psi);$$

setting $\Delta p \ll P \sim P_0$, we obtain

$$P' = \sqrt{\mathbf{P}^2 - 2\mathbf{P}\Delta\mathbf{p} + \Delta\mathbf{p}^2} = P - \mathbf{P}\Delta\mathbf{p}/P_0 + \Delta\mathbf{p}^2/2P_0 - (\mathbf{P}\Delta\mathbf{p})^2/2P_0^3.$$

Setting this value in (12) and squaring, we find

$$\frac{(\delta p)^2}{2} = \frac{1}{2p^2}\left(\frac{m}{\mu}\right)^2\left[\frac{P-P_0}{P_0}\left(\frac{\Delta\mathbf{p}^2}{2} - \mathbf{P}\Delta\mathbf{p}\right) + \frac{(\mathbf{P}\Delta\mathbf{p})^2}{2P_0^2}\right]^2.$$

We must now average this expression over the roton distribution. The average values of $P - P_0$ and $\mathbf{P}\Delta\mathbf{p}$ fall to zero in this process. Indicating by χ the angle between $\mathbf{P}$ and $\Delta\mathbf{p}$, we obtain:

$$\frac{1}{2}\langle(\delta p)^2\rangle_{\mathrm{r}} = \frac{1}{2p^2}\left(\frac{m}{\mu}\right)^2\left\{p^4(1-\cos\psi)^2\left[\frac{\langle(P-P_0)^2\rangle_p}{P_0^2} - \langle\cos^4\chi\rangle_{\mathrm{r}}\right] + 2p^2(1-\cos\psi)\langle(p-p_0)^2\rangle_{\mathrm{r}}\langle\cos^2\chi\rangle_{\mathrm{r}}\right\}.$$

It is easy to obtain

$$\langle(P-P_0)^2\rangle_{\mathrm{r}} = \mu kT; \quad \langle\cos^2\chi\rangle_{\mathrm{r}} = 1/3; \quad \langle\cos^4\chi\rangle_{\mathrm{r}} = 1/5.$$

Integrating over the angle ψ we obtain finally* (in view of the inequality $\mu kT \ll P_0^2$):

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*A similar computation for the collision of electrons with the ions of a plasma yields

$$A_2^{\mathrm{i}} = \frac{v}{l}\left(\frac{m}{\mu}\right)^2\left(MkT + \frac{2p^2}{3}\right).$$

The first term was found by Davydov, and is the dominant one; the second becomes important only for extremely strong electric fields.

$$A_2^{\mathrm{r}} = \frac{1}{2}\int \langle(\delta p)^2\rangle_{\mathrm{r}} \frac{v}{l}\frac{do}{4\pi} = \frac{v}{l}\left(\frac{m}{\mu}\right)^2\left(\frac{\mu kT}{3} + \frac{2p^2}{15}\right). \qquad (13)$$

Substituting in (4) and (5) the values

$$A_2 = A_2^{\mathrm{ph}} + A_2^{\mathrm{r}}, \quad B = B^{\mathrm{ph}} + B^{\mathrm{r}},$$

we obtain the solution of the problem under consideration.

For the sake of brevity, we shall discuss here only a few simple limiting formulas, for pure phonon and pure roton gases.

For the phonon gas in extremely weak fields, for which $eE\lambda \ll (kT)^{3/2}/(ms^2)^{1/2}$, it is possible to take $e^2E^2 \ll A_2B$. The usual Maxwellian distribution is then obtained for f_0. Computing B^{ph} from Eqs. (9) and (10), we can take $sp/kT \ll 1$, and obtain the value

$$B^{\mathrm{ph}} = pkT/\lambda m^2 s^2, \qquad (14)$$

with the aid of which we obtain from Eq. (5) the current density

$$j = {}^2/_3\sqrt{2/\pi}\, ne^2 s^2 m^{1/2}\lambda E\,(kT)^{-3/2}, \qquad (15)$$

in this case depending linearly upon the electric field intensity.

In the region for which the electric field intensity obeys the inequalities

$$(kT)^{3/2}/(ms^2)^{1/2} \ll eE\lambda \ll (kT)^{5/2}/(ms^2)^{3/2},$$

we have

$$A_2B \ll e^2E^2,$$

and Eq. (14) for B^{ph} is still applicable. In this case we obtain for the distribution function the value

$$f_0 \sim \exp\{-3p^4kT/4m^3s^2(eE\lambda)^2\},$$

and the current density turns out to be proportional to $E^{1/2}$:

$$j = \left(\sqrt{2\pi}/3^{3/4}\,\Gamma({}^3/_4)\right) ne^{3/2}s^{3/2}m^{1/4}\lambda^{1/2}E^{1/2}/(kT)^{3/4}. \qquad (16)$$

If $eE\lambda \gg (kT)^{5/2}/(ms^2)^{3/2}$, then* $sp/kT \gg 1$ and we obtain

$$B^{\mathrm{ph}} = 4p^2/5\lambda m^2 s.$$

The expression for f_0 for this case includes in the exponent the sixth power of the momentum

$$f_0 \sim \exp\{-8p^6/25\, m^3kT(eE\lambda)^2\},$$

and the current density is proportional to $E^{1/3}$:

$$j = \left(5^{1/3}\,\Gamma({}^1/_6)/6\sqrt{\pi}\right) ne^{4/3}s\lambda^{1/3}E^{1/3}/(kT)^{1/3}. \qquad (17)$$

For the case of a pure roton gas we shall also discuss three ranges of magnitudes for the electric field. For fields fulfilling the condition $eE\lambda \ll (m/\mu)^{1/2}kT$ we have $e^2E^2 \ll BA_2$, and the Maxwellian distribution function is again obtained for f_0. Calculation of the current density yields a linear dependence upon field intensity

$$j = {}^2/_3\sqrt{2/\pi}\, ne^2E/\sigma N_{\mathrm{r}}(mkT)^{1/2}. \qquad (18)$$

In fields for which the inequality

$$(m/\mu)^{1/2}kT \ll eEl \ll (\mu/m)^{1/2}kT,$$

is fulfilled we have $e^2E^2 \gg BA_2^{\mathrm{r}}$, and in A_2^{r} it is sufficient to stop with the first term $A_2^{\mathrm{r}} = mpkT/3\mu\ell$. Using this value we readily obtain the distribution function

$$f_0 \sim \exp\{-p^4/4m\mu(eEl)^2\}$$

---

*Taking the average value of p over the distribution.

and the electric current density

$$j = \left(\sqrt{2\pi}/3\,\Gamma(^3/_4)\right) ne^{3/2}E^{1/2}/\sigma^{1/2}N_r^{1/2}(m\mu)^{1/4}. \tag{19}$$

Finally, for $eE\ell \gg (\mu/m)^{1/2}kT$ the second term in the expression for A_2^r predominates, and $A_2^r = 2p^3m/15\ell\mu^2$.

In this case* we obtain for the distribution function the value

$$f_0 \sim \exp\{-p^6/15\,m\mu^2kT\,(eEl)^2\}$$

and for the current density

$$j = \frac{2\Gamma(^1/_3)}{3^{7/6}5^{1/6}\sqrt{\pi}}\,\frac{ne^{5/3}E^{2/3}}{(N_r\sigma)^{2/3}(m\mu^2kT)^{1/6}}. \tag{20}$$

These formulas contain the unknown parameters λ, σ, the effective mass m, and also the number of electrons per unit volume n. If it were possible to perform measurements in all three ranges of electric field dependence, all of these unknown parameters could be determined. It should be remembered, however, that the magnitudes of the parameters can cause overlapping of the phonon and roton regions and thereby complicate somewhat the interpretation of the experimental results. For example, the condition $(\mu/m)^{1/2}kT \ll eE\ell$ corresponds to $eE\ell \gg \Delta$, and the creation of several rotons, as well as phonons, can accompany the scattering of electrons by a roton. In this case the dependence (17) will be observed in place of (20). Experimental investigation of the behavior of particles of small effective mass in He II would be of definite interest.

I would like to express my deep gratitude to L. D. Landau and I. M. Khalatnikov for their helpful discussions of this work.

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STATIONARY CONVECTION IN A PLANE LIQUID LAYER NEAR THE CRITICAL HEAT TRANSFER POINT

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Thermal convection which arises above the limit of stability is investigated for a plane liquid layer. A dependence of the amplitude of motion on a parameter which characterizes the departure from critical conditions has been obtained. Possible symmetry of the flow is discussed.

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*An analogous situation can in principle arise for the scattering of electrons by heavy atoms. Here an $\sim E^{2/3}$ dependence for the current is also obtained, but with different numerical coefficients; this is understandable, since A_2 is different for ions. In this case, however, the electrons have sufficient energy for ionization, which must be taken into account.

HEAT transfer in a plane liquid layer with constant temperature gradient was studied experimentally by Benard,[1] who showed that for certain values of a downward-directed temperature gradient pure thermal conduction is replaced by convection with certain characteristics in a stationary mode; the flow is divided among regular vertical hexagonal prisms and thus possesses hexagonal symmetry. The critical gradient at which convection occurs has been calculated in several papers (see Ref. 2), but the utilization of linearized hydrodynamic equations permitted the determination of only the magnitude of the periods. The determination of the symmetry and amplitude of the motion for supercritical gradients requires study of the nonlinear equations. The present paper presents the results of such an investigation for modes close to the critical value.

A plane horizontal liquid layer of thickness h is bounded above and below by two planes between which a constant temperature difference is maintained. The boundary conditions can be taken in three forms: (a) two rigid planes, (b) one rigid plane and one free surface, and (c) two free surfaces. The last case is distinguished by the simplicity of its equations. Benard's experiments were performed under conditions (b). We shall write the basic equations of stationary convection as follows (see Ref. 3):

$$(\mathbf{v}\nabla)\,\mathbf{v} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \beta\mathbf{g}\theta + \nu\Delta\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0, \quad (\mathbf{v}\nabla\theta) - Av_z = \chi\Delta\theta. \tag{1}$$

Here p and θ are small deviations, due to convection, from the equilibrium pressure P and temperature T:

$$P = p_0 - \rho g z + p; \quad T = -Az + T_0 + \theta,$$

v is the velocity of the motion; $\beta = -\rho^{-1}(\partial\rho/\partial T)_p$. The boundary conditions at the rigid surface are:

$$\theta = 0, \quad v_z = 0; \quad \partial v_z/\partial z = 0 \quad (v_x = v_y = 0), \tag{1a}$$

and at the free surface:

$$\theta = 0; \quad v_z = 0; \quad \partial^2 v_z/\partial z^2 = 0 \quad (\sigma_{xz} = \sigma_{yz} = 0). \tag{1b}$$

For convenience we shall retain the vector notation only for variables in the horizontal plane $\mathbf{x} = (x, y)$, separating the z coordinate; for all quantities we shall employ a Fourier expansion in these variables:

$$\mathbf{v}(x, y, z) = \sum_k \mathbf{v}_k(z)\, e^{i\mathbf{kx}} \tag{2}$$

and similarly for v_z, p and θ. By virtue of the boundary conditions $v_{0z} = 0$. The velocity $\mathbf{v}_k(z)$ is resolved into components which are parallel and perpendicular to the wave vector **k**:

$$\mathbf{v}_k = \widetilde{\mathbf{v}}_k + \mathbf{k}\,(\mathbf{k}\mathbf{v}_k)/k^2.$$

The second and third equations of (1) become

$$i\,(\mathbf{k}\mathbf{v}_k) + dv_{kz}/dz = 0, \tag{3}$$

$$\chi\left(\frac{d^2}{dz^2} - k^2\right)\theta_k + Av_{kz} = \sum_{k'+k''=k}\left\{i\,(\mathbf{k}\mathbf{v}_{k'})\,\theta_{k''} + \frac{d}{dz}(v_{k'z}\,\theta_{k''})\right\}. \tag{4}$$

Eliminating the pressure in the first equation,

$$\frac{1}{\rho}p_k = \frac{\nu}{k^2}\left(\frac{d^2}{dz^2} - k^2\right)\frac{dv_{kz}}{dz} + \frac{i}{k^2}\sum_{k'+k''=k}\left\{i\,(\mathbf{k}\mathbf{v}_{k'})\,(\mathbf{k}\mathbf{v}_{k''}) + \frac{d}{dz}(v_{k'z}(\mathbf{k}\mathbf{v}_{k''}))\right\},$$

we also obtain

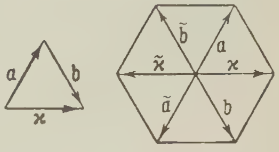
$$\begin{aligned} -\nu\left(\frac{d^2}{dz^2} - k^2\right)^2 v_{kz} + k^2\beta g\theta_k = \sum_{k'+k''=k}\Big\{k^2\Big[i\,(\mathbf{k}\mathbf{v}_{k'})\,v_{k''z} \\ + \frac{d}{dz}(v_{k'z}v_{k''z})\Big] + i\frac{d}{dz}\Big[i\,(\mathbf{k}\mathbf{v}_{k'})\,(\mathbf{k}\mathbf{v}_{k''}) + \frac{d}{dz}(v_{k'z}\,(\mathbf{k}\mathbf{v}_{k''}))\Big]\Big\}, \end{aligned} \tag{5}$$

$$\nu\left(\frac{d^2}{dz^2} - k^2\right)\widetilde{\mathbf{v}}_k = \sum_{k'+k''=k}\left\{i\,(\mathbf{k}\mathbf{v}_{k'})\left[\mathbf{v}_{k''} - \frac{\mathbf{k}\,(\mathbf{k}\mathbf{v}_{k''})}{k^2}\right] + \frac{d}{dz}\left(v_{k'z}\left[\mathbf{v}_{k''} - \frac{\mathbf{k}\,(\mathbf{k}\mathbf{v}_{k''})}{k^2}\right]\right)\right\}. \tag{6}$$

Finally, we give the equations for the Fourier components with k = 0:

$$\chi \frac{d^2\theta_0}{dz^2} = \frac{d}{dz}\sum_{k'}(v_{k'z}\theta_{-k'}); \ \frac{d}{dz}\left\{\nu\frac{d\mathbf{v}_0}{dz} - \sum_{k'} v_{k'z}\mathbf{v}_{-k'}\right\} = 0; \quad \frac{1}{\rho}\frac{dp_0}{dz} = \beta g\theta_0 - \frac{d}{dz}\sum_{k'}(v_{k'z}v_{-k'z}).$$

Before proceeding to study the equations which have been derived we shall determine the vectors **k** of the Fourier series (2). As was shown by the investigation of stability in Ref. 2, when the critical temperature gradient is attained there arises convective motion of the form $f(z)e^{i\mathbf{kx}}$ with a certain value $k = \kappa_0$ which depends on the experimental conditions (a, b or c). In (2), and similarly in the expansions for v_z and θ, we shall call $v_{\mathbf{k}z}$ with $k = \kappa_0$ the principal term. It is easy to determine all such vectors of the flow. Indeed, as can be seen from (3)–(6), corrections to the principal terms with $k = \kappa_0$ will consist of terms with wave vectors which are again equal in magnitude to κ_0. Therefore, if (2) contains a term with the wave vector $\kappa(|\kappa| = \kappa_0)$ there must also be terms with the vectors **a** and **b** $(a = b = \kappa_0)$, thus forming the equilateral triangle of Fig. a. It is clear that there are altogether six such vectors $\kappa, \mathbf{a}, \mathbf{b}, \tilde{\kappa}, \tilde{\mathbf{a}}, \tilde{\mathbf{b}}$, which are represented together in Fig. b. The Fourier components which are thus designated, for all quantities, comprise the set of principal terms of the solution; the remaining terms correspond to wave vectors representing various sums of the six principal vectors and will be small for temperature gradients near the critical value.



a b

We shall write the principal terms in the form

$$v_{iz} = X_i f(z); \quad \theta_i = X_i\varphi(z),$$

where[2]

$$\{(d^2/dz^2 - \varkappa_0^2)^3 + (A\beta g\varkappa_0^2/\chi\nu)\} f(z) = 0; \ \varphi(z) = (\nu/\varkappa_0^2\beta g)(d^2/dz^2 - \varkappa_0^2)^2 f(z). \tag{7}$$

It will be sufficient hereinafter to confine ourselves to terms of the third order in X, i.e., to take into account all triple combinations of the principal vectors. This means that all quantities $v_{\mathbf{k}z}$ and $\theta_{\mathbf{k}}$ can be obtained up to terms of the second order in X; in other words, in addition to the corrections $\delta v_{\kappa z}$ and $\delta\theta_\kappa$ (and similarly for **a**, **b** etc.) to the expressions in (7), it is necessary to obtain the Fourier components for the velocities and temperature corresponding to wave vectors each of which is a sum of two principal vectors; for example $\mathbf{f} = \boldsymbol{\kappa} + \mathbf{b}, 2\boldsymbol{\kappa}, \mathbf{o}$ etc. We note that by virtue of the equation of continuity (3) expressions of the form $(\mathbf{qv}_\ell)$ in (4) and (5) can be rewritten as

$$(\mathbf{qv}_l) = (\mathbf{q}\tilde{\mathbf{v}}_l) + i(\mathbf{ql})l^{-2}dv_{lz}/dz.$$

From (6) it follows, to terms of the second order, that

$$\nu\left(\frac{d^2}{dz^2} - l^2\right)(\mathbf{q}\tilde{\mathbf{v}}_l) = i\sum_{k'+k''=l}\left\{\left[\frac{(\mathbf{qk}'')}{k''^2} - \frac{(\mathbf{ql})(\mathbf{lk}'')}{l^2k''^2}\right]\left[\frac{d}{dz}\left(v_{k'z}\frac{dv_{k''z}}{dz}\right) - \frac{(\mathbf{lk}')}{k'^2}\frac{dv_{k'z}}{dz}\frac{dv_{k''z}}{dz}\right]\right\}.$$

Substituting the expressions in (7) we see that for all required $\boldsymbol{\ell}$ and **q** the right-hand side vanishes. Second-order terms in $(\mathbf{q}\tilde{\mathbf{v}}_\ell)$ vanish so that in (4) and (5) we can write

$$(\mathbf{qv}_l) \to i(\mathbf{ql})l^{-2}dv_{lz}/dz.$$

For brevity we introduce the notation

$$D_\varkappa = (d^2/dz^2 - \varkappa_0^2).$$

From (4) and (5), including third-order terms in X, we obtain

$$D_k^3\theta_k + \frac{k^2A\beta g}{\chi\nu}\theta_k = \frac{1}{\chi}\sum_{k'+k''=k} D_k^2\left\{\frac{d}{dz}(v_{k'z}\theta_{k''}) - \frac{(\mathbf{kk}')}{k'^2}\frac{dv_{k'z}}{dz}\theta_{k''}\right\} + \frac{A}{\chi\nu}\sum_{k'+k''=k}\left\{k^2\left[\frac{d}{dz}(v_{k'z}v_{k''z}) - \frac{(\mathbf{kk}')}{k'^2}\frac{dv_{k'z}}{dz}v_{k''z}\right]\right.$$
$$\left. + \frac{(\mathbf{kk}'')}{k''^2}\frac{d}{dz}\left[\frac{(\mathbf{kk}')}{k'^2}\frac{dv_{k'z}}{dz}\frac{dv_{k''z}}{dz} - \frac{d}{dz}\left(v_{k'z}\frac{dv_{k''z}}{dz}\right)\right]\right\}. \tag{8}$$

The magnitude of the dimensionless parameter $\gamma = A\beta gh^4/\chi\nu$ determines the possibility of the existence of stationary convective motion. The critical value γ_0 at which instability arises[2] is for two rigid surfaces ≈ 1708, for one free surface ≈ 1100 and for two free surfaces ≈ 657. We shall write (8) for the κ component as follows:

$$D_\varkappa^3\theta_\varkappa + \gamma_0\theta_\varkappa = -\Delta\gamma\theta_\varkappa + R_\varkappa(\theta^3)$$

where $R_\kappa(\theta^3)$ is the right-hand side of (8) and $\Delta\gamma = \gamma - \gamma_0$. The homogeneous equation has the solution $\varphi(z)$, with γ_0 possessing the same meaning. The condition

$$\int f\{R_\varkappa(\theta^3) - \Delta\gamma\theta_\varkappa\}\,dz = 0, \tag{9}$$

which is required for the solution of (8), leads to the determination of the amplitude of motion X. This condition has the form

$$\frac{\varkappa_0^2\Delta\gamma}{h^4}X_\varkappa\int f\varphi dz = \frac{1}{\chi}\int(D_\varkappa^2 f)\sum_{k'+k''=\varkappa}\left\{\frac{d}{dz}(v_{k'z}\theta_{k''}) - \frac{(\varkappa\mathbf{k}')}{k'^2}\frac{dv_{k'z}}{dz}\theta_{k''}\right\}dz + \frac{A\varkappa_0^2}{\chi\nu}\int f\sum_{k'+k''=\varkappa}\left\{\frac{d}{dz}(v_{k'z}v_{k''z}) - \frac{(\varkappa\mathbf{k}')}{k'^2}\frac{dv_{k'z}}{dz}v_{k''z}\right\}dz$$

$$+\frac{A}{\chi\nu}\int\frac{df}{dz}\sum_{k'+k''=\varkappa}\left\{\frac{(\mathbf{k}\mathbf{k}'')}{k''^2}\left[\frac{d}{dz}\left(v_{k'z}\frac{dv_{k''z}}{dz}\right) - \frac{(\mathbf{k}\mathbf{k}')}{k'^2}\frac{dv_{k'z}}{dz}\frac{dv_{k''z}}{dz}\right]\right\}dz.$$

Second-order terms in X drop out, as is shown by substitution of (7) into the right-hand side;* for the purpose of calculating third-order terms it is necessary to obtain the corrections to v_z and θ. Such corrections appear for the following quantities (see Fig. b):

$$\delta\theta_a = X_{\tilde{b}}X_\varkappa\xi_1(z);\ \delta\theta_b = X_{\tilde{a}}X_\varkappa\xi_1(z);\ \delta\theta_f = X_\varkappa X_b\xi_2(z);\quad \delta\theta_g = X_aX_\varkappa\xi_2(z);\ \delta\theta_{2\varkappa} = X_\varkappa^2\xi_3(z);$$

$$\delta\theta_0 = (|X_a|^2 + |X_a|^2 + |X_\varkappa|^2)\,\xi_4(z).\ \ \delta v_{az} = X_{\tilde{b}}X_\varkappa\eta_1(z);\ \delta v_{bz} = X_{\tilde{a}}X_\varkappa\eta_1(z);\ \delta v_{fz} = X_\varkappa X_b\eta_2(z);$$

$$\delta v_{gz} = X_\varkappa X_a\eta_2(z);\ \delta v_{2\varkappa} = X_\varkappa^2\eta_3(z).$$

Using these designations (9) becomes

$$\frac{\chi\nu\Delta\gamma}{\beta g\hbar^4}\int(fD^2f)\,dz = (|X_a|^2 + |X_b|^2)\int\left\{\frac{A}{2\nu}[Df'f(\eta_1+\eta_2) + 2f'Df\eta_1] - fD^2f'(\xi_1+\xi_2+\xi_4) - \frac{D^2f}{2}f'(\xi_1-\xi_2+2\xi_4)\right\}dz$$

$$+|X_\varkappa|^2\int\left\{\frac{A}{2\nu}[f'''f - f''f']\,\eta_3 + \left[\xi_3(D^2ff' - D^2f'f) - \xi_4\frac{d}{dz}(fD^2f)\right]\right\}dz$$

where primes denote differentiation with respect to z. Two other equations are then obtained directly by the substitutions: κ→a, a→b, b→κ, or κ→b, a→κ, b→a.

The solution of these equations is obviously $|X_\kappa| = |X_a| = |X_b|$; the amplitude of convective motion is then proportional to the square root of the parameter Δ_γ which characterizes the supercritical heat transfer mode. But our equations contain only the absolute value of X so that the phase relation cannot be determined. It is clear how this has come about: when we write (9) for θ_κ, say, we must select in R all combinations corresponding to any three principal vectors whose vector sum is x. It is easily seen from the figure that any such combination must be of the form $X_\kappa|X_i|^2$ so that X_κ is cancelled and (10) contains only the square of the modulus of X. Fourth-order terms in X in (9) obviously drop out. In the fifth order it is possible to obtain a combination which is not proportional to X_κ: the sum $2\mathbf{a} + 2\mathbf{b} + \tilde{\boldsymbol{\kappa}}$ in the right-hand side gives a term which is proportional to $X_\kappa \sim X_a^2X_b^2$. Writing $X = |X|e^{i\delta}$, we obtain because of the reality of the coefficients the following phase relation:

$$\delta_a + \delta_b - \delta_\varkappa = 0,\ \pi/2.$$

In the first case it is possible to make all phases equal to zero by shifting the coordinate origin. The principal term in the solution becomes

$$v_z = B\left(\frac{\chi}{\nu}\right)f(z)\sqrt{\frac{\Delta\gamma}{\gamma}}\left\{\cos\varkappa_0 y + \cos\frac{\varkappa_0}{2}(y+\sqrt{3}x) + \cos\frac{\varkappa_0}{2}(y-\sqrt{3}x)\right\} \tag{11a}$$

where the y axis is parallel to κ. This solution possesses hexagonal symmetry; in the (x, y) plane it represents a periodic structure of regular hexagonal prisms of the length $4\pi/\kappa_0$ along each side of the base; the liquid flows upward at the center of each prism and downward along the walls.[2]

In the second case, with a suitable choice of the coordinate origin, the solution can be written as follows:

$$v_z = B\left(\frac{\chi}{\nu}\right)f(z)\sqrt{\frac{\Delta\gamma}{\gamma}}\left\{\sin\frac{\varkappa_0}{2}(y+\sqrt{3}x) + \sin\frac{\varkappa_0}{2}(y-\sqrt{3}x) - \sin\varkappa_0 y\right\}. \tag{11b}$$

This flow permits three-fold rotations around the vertical axis (the z axis) and is symmetric with respect to the x axis, but changes sign with the substitution y → −y. In the horizontal plane it breaks up into alternate equilateral triangles of side length $4\pi/\sqrt{3}\kappa_0$, in each of which the liquid flows upward or downward, respectively.

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*For symmetrical boundary conditions this is clear from simple considerations, since the solution of f(z) is symmetrical with respect to z = 0.

A decision as to which of the two types of flow will occur in reality could be reached theoretically only by investigating fifth-order terms in X in the basic equations of convection, but in actuality it would be difficult to do this. For a liquid layer on a rigid plate experiment[1] favors the first type of flow.

The equations for the ξ functions are

$$\left[D^3+\frac{\varkappa_0^2\gamma}{h^4}\right]\xi_1=\frac{2\nu}{\varkappa_0^2\beta g\chi}\left\{D^2\left(fD^2f'+\frac{f'}{2}D^2f\right)-\frac{A\beta g\varkappa_0^2}{\nu^2}\left(fDf'+\frac{f'}{2}Df\right)\right\};$$

$$\left[(D-2\varkappa_0^2)^3+\frac{3\varkappa_0^2\gamma}{h^4}\right]\xi_2=\frac{2\nu}{\varkappa_0^2\beta g\chi}\left\{(D-2\varkappa_0^2)^2\left(fD^2f'-\frac{f'}{2}D^2f\right)-\frac{3A\beta g\varkappa_0^2}{2\nu^2}fDf'\right\}; \tag{12}$$

$$\left[(D-3\varkappa_0^2)^3+\frac{4\varkappa_0^2\gamma}{h^4}\right]\xi_3=\frac{\nu}{\varkappa_0^2\beta g\chi}\left\{(D-3\varkappa_0^2)^2(fD^2f'-f'D^2f)+\frac{2A\beta g\varkappa_0^2}{\nu}(f'f''-ff''')\right\};\quad d^2\xi_4/dz^2=(2\nu/\varkappa_0^2\beta g\chi)\,d(fD^2f)/dz,$$

and the expressions for the η functions are

$$A\eta_1=-\chi D\xi_1+2(f\varphi'+{}^1/_2\,\varphi f');\qquad A\eta_2=-\chi(D-2\varkappa_0^2)\xi_2+2(f\varphi'-{}^1/_2\,\varphi f');\quad A\eta_3=-\chi(D-3\varkappa_0^2)\xi_3+(f\varphi'-f'\varphi). \tag{13}$$

The calculation of these quantities subject to the boundary conditions 1a and 1b would be very laborious for the general case. The problem becomes simpler only in the somewhat hypothetical case of a liquid with two free surfaces. We present the results in so far as they are of a general character.

In this case the critical value is $\gamma_0 = 27\pi^4/4$, $\kappa_0^2 = \pi^2/2h^2$ and $f(z)$, which determines the dependence of the vertical velocity component on height, can be taken in the form

$$f(z)=\cos\frac{\pi z}{h}\left(\text{or}\quad \varphi(z)=\frac{9\nu\pi^2}{2\beta gh^2}\cos\frac{\pi z}{h}\right).$$

Equations (12) and (13) can be solved simply and give for ξ and η

$$\left.\begin{aligned}\xi_1&=\frac{9\nu\pi[9+\chi/\nu]}{52\chi\beta gh}\\ \xi_2&=\frac{9\nu\pi[121+27\chi/\nu]}{2500\chi\beta gh}\\ \xi_4&=\frac{9\nu\pi}{4\chi\beta gh}\end{aligned}\right\}\times\sin\frac{2\pi z}{h},\quad \left.\begin{aligned}\eta_1&=\frac{h[1+3\chi/\nu]}{26\pi\chi}\\ \eta_2&=\frac{9h[3+11\chi/\nu]}{1250\pi\chi}\end{aligned}\right\}\times\sin\frac{2\pi z}{h},$$

$$\xi_3=0.$$

By substituting these expressions in (10) we can determine X. For example, the temperature distribution for the first type of flow is given by

$$\theta(xyz)=\frac{4Ah}{3\pi}\sqrt{\frac{\Delta\gamma}{\gamma}}\frac{\cos(\pi z/h)}{\sqrt{1.35+0.08\chi/\nu+0.14\chi^2/\nu^2}}\left\{\cos\frac{\pi y}{2h}+\cos\frac{\pi}{4h}(y+\sqrt{3}x)+\cos\frac{\pi}{4h}(y-\sqrt{3}x)\right\} \tag{14}$$

The development of convection is conveniently characterized by the ratio of the maximum temperature change in the (x, y) plane, $T_{\max} - T_{\min}$, to the vertical temperature difference $T_1 - T_2 = Ah$:

$$\frac{T_{\max}-T_{\min}}{T_1-T_2}=\frac{4}{\pi}\sqrt{\frac{\Delta\gamma}{\gamma}}\left(1.35+0.08\,\frac{\chi}{\nu}+0.14\,\frac{\chi^2}{\nu^2}\right)^{-1/2}.$$

We note in conclusion that the form of the dependence of the coefficient in (14) on the Prandtl number is the same for any boundary conditions. This results from the fact that the solution of the linear equation $f(z) \equiv f(z/h)$ depends only on the critical value γ_0, and also on how (12), (13) and (10) contain the different dimensional quantities. Thus the strength of stationary thermal convection in a plane liquid layer for modes near the critical point is proportional to the square root of the parameter which characterizes supercriticality, whereas dependence on the Prandtl number appears in a denominator as the square root of a polynomial of not higher than the second degree. The flow belongs to one of the two types of symmetry represented by (11a) and (11b).

The author is indebted to Academician L. D. Landau for valuable suggestions and assistance.

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ON THE THEORY OF A POSITIVE COLUMN IN A LONGITUDINAL MAGNETIC FIELD

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This article considers the contraction of the positive column in a gas discharge located in an external magnetic field directed along the discharge axis.

IT has been shown experimentally[1,2] that in the positive column of a gas discharge in an external magnetic field there occurs a redistribution of the current density through the cross section. The current density increases at the axis, and decreases at the walls. A generalization[3] of Schottky's ambipolar diffusion theory to a column in a longitudinal magnetic field failed to explain the observed phenomena. Fabrikant[4] feels that this is due to insufficiencies in the boundary condition used by Tonks. Similar conclusions can be reached on the basis of the recent work of Engel and Bicerton.[5] Since the behavior of a plasma in a magnetic field is of great interest, we feel that a theoretical treatment of the current-density redistribution due to a longitudinal magnetic field is of value.

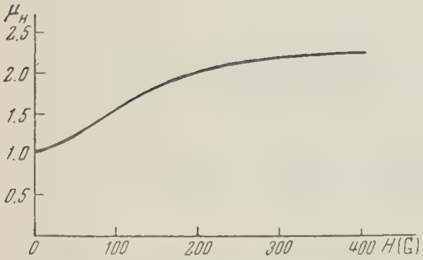

FIG. 1. The root of the transcendental equation (6) as a function of the magnetic field. The graph is for argon, with p = 0.0037 mm Hg, I = 300 ma, and R = 2.2 cm.

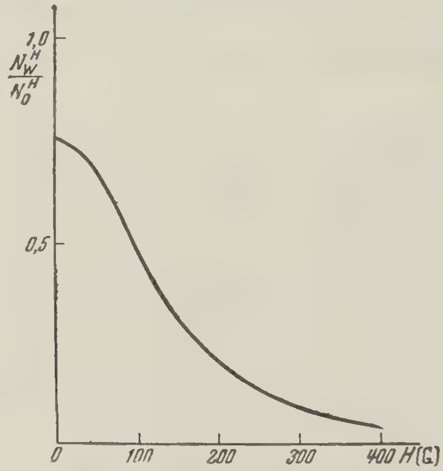

FIG. 2. Wall-to-axis concentration ratio as a function of the magnetic field in argon with the same parameters as those of Fig. 1.

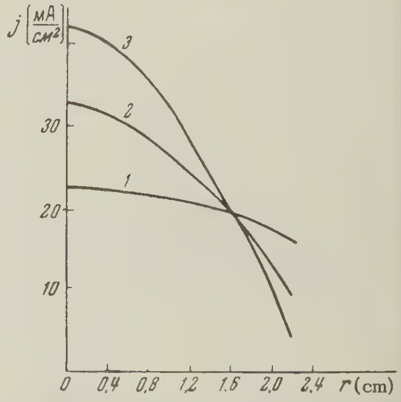

FIG. 3. Current density distribution over the cross section (with the magnetic field as a parameter). The calculations are performed for argon with the same conditions as in Fig. 1. Curve 1 — H = 0; 2 — H = 167 gauss; 3 — H = 365 gauss.

1. As in the previous work of Terletskii and the author[6] we start our consideration of the positive-column plasma with the hydrodynamic equations for ideal electron and ion gases uniformly distributed through the cross section in a neutral gas. The magnetic field is accounted for* by adding to the right side of the equations of motion the forces exerted by the magnetic field on the moving charged particles. We consider the radially symmetric case with the field directed along the Z axis. Then on the assumption of a stationary state for ambipolar diffusion,† we obtain

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*For sufficiently low currents one may neglect the magnetic field due to the current, compared with the external field.

†The equations referring to flow along the axis are not included. They are not changed by the addition of the magnetic field.

$$\Theta_e \frac{\partial N_e}{\partial r} + \frac{eN_e}{m_e} E_r + \alpha_e N_e v_{er} + \frac{eN_e}{m_e c} H v_{e\varphi} = 0, \quad \alpha_e N_e v_{e\varphi} - \frac{eN_e}{m_e c} H v_{er} = 0, \ \frac{1}{r} \frac{\partial}{\partial r}(rN_e v_{er}) = Z^H N_e,$$

$$\Theta_p \frac{\partial N_p}{\partial r} - \frac{eN_p}{m_p} E_r + \alpha_p N_p v_{pr} - \frac{eN_p}{m_p c} H v_{p\varphi} = 0, \ \alpha_p N_p v_{p\varphi} + \frac{eN_p}{m_p c} H v_{pr} = 0, \ \frac{1}{r} \frac{\partial}{\partial r}(rN_p v_{pr}) = Z^H N_e, \tag{1}$$

where N_e and N_p are the charged-particle concentrations, H is the magnetic field strength, m_e, v_{er}, $v_{e\varphi}$, α_e, and Θ_e are the mass, velocity components, coefficient of friction, and temperature (in ergs) of the electrons (and similarly for the positive ions), and Z^H is the number of ionization events per second per electron. After eliminating the velocities and field from Eq. (1), the assumption of quasi-neutrality $N_e \approx N_p = N$ for the plasma leads to the equation

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial N^H}{\partial r}\right) + \frac{Z^H}{D_a^H} N^H = 0, \tag{2}$$

for the charged-particle distribution through the cross section, where

$$D_a^H = \frac{D_e b_p + D_p b_e}{b_p \{1 + (\omega_e / \alpha_e)^2\} + b_e \{1 + (\omega_p / \alpha_p)^2\}} \tag{3}$$

is the ambipolar diffusion constant in the longitudinal magnetic field, ω_e and ω_p are the Larmor frequencies for the electrons and ions, and b_e and b_p are the mobilities of the electrons and ions in the absence of the magnetic field.

The boundary condition proposed by Schottky[7] for the walls does not lead to the current distribution found in experiment,* so that we have chosen another condition, mentioned by van de Groot and used by Granovskii[8] in de-ionization theory. If one neglects the creation of particles in a layer at the walls whose thickness is equal to the mean free path,[5] this boundary condition can be written

$$-D_a^H dN^H / dr\,|_{r=R} = {}^1\!/_2\, N^H c_p - D_p^H \partial N^H / dr\,|_{r=R}, \tag{4}$$

where c_p is the thermal velocity of the positive ions.

2. A solution of Eq. (2) with a finite concentration on the axis is of the form

$$N^H = N_0^H J_0\left(\sqrt{Z^H / D_a^H}\, r\right), \tag{5}$$

where N_0^H is the charged-particle concentration on the axis in the presence of the magnetic field. If one accounts for the fact that $D_a^H \gg D_p^H$, Eq. (4) leads to the transcendental equation †

$$2D_a^H \mu^H / c_p R = J_0(\mu_H) / J_1(\mu_H), \tag{6}$$

which defines $\mu_H = \sqrt{Z^H / D_a^H}\, R$. Graphical solution makes it possible to determine μ_H as a function of the magnetic field (Fig. 1).

3. In order to compare the calculated results with experiment, let us consider the charged-particle concentration and current density in the tube cross section. On the assumption that the total current is the same with and without a magnetic field, as well as that the fraction of the total energy expended on ionization is field independent, we obtain the following relation between the concentrations on the axis of discharge:

$$N_0^H / N_0 = \sqrt{D_a / D_a^H}\, J_1/(\mu) / J_1(\mu_H). \tag{7}$$

This equation makes it possible to express the charged particle distribution in the magnetic field in terms of the concentration at the axis in its absence, namely

$$N^H = N_0 \sqrt{\frac{D_a}{D_a^H} \frac{J_1(\mu)}{J_1(\mu_H)}}\, J_0\left(\sqrt{\frac{Z^H}{D_a^H}}\, r\right). \tag{8}$$

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*It should be noted that for a stationary current-density distribution, the concentration increases over the whole cross section.

†For small values of the mean free path perpendicular to the tube axis, Eq. (6) can be reduced to a linear equation for μ_H. In general one cannot obtain an explicit expression for μ_H as a function of the discharge parameters and the magnetic field, and it was therefore necessary to perform a graphical solution.

Equation (8) can be used to calculate the wall-to-axis concentration ratio as a function of the magnetic field:

$$N_W^H/N_0^H = J_0(\mu_H), \tag{9}$$

where N_W^H is the charged-particle concentration at the wall. The graph of this function shown in Fig. 2 gives satisfactory qualitative agreement with experimental results for helium.*

Assuming that the current passing through the column is due only to the drift motion of the electrons and that the longitudinal gradient is determined by the equation

$$E_Z^H = (\mu_H/R)\sqrt{D_a^H/\alpha}, \tag{10}$$

we obtain the following expression for the current density distribution through the cross section:

$$j_H = \frac{I}{2\pi R^2}\,\frac{\mu_H}{J_1(\mu_H)}\,J_0\left(\frac{\mu_H}{R}r\right), \tag{11}$$

Here I is the total current passing through the cross section of the conductor. Curves of the current density distribution (Fig. 3) show that the magnetic field redistributes the current, increasing its density at the axis and decreasing it at the wall. This is in qualitative agreement with the observations of Reikhrudel' and Spivak.[1] The difference between the theoretical and experimental results can be due to uncertainties, but it would seem that it is partly caused also by the neglect of cascade processes and the magnetic-field dependence of such parameters as the electron temperature. The calculations performed indicate the significant role that processes taking place at the walls of the discharge tube play in the contraction of the column. This role is not the same at different gas pressures, and should be decreased by a pressure rise for a given magnetic field strength. The disappearance of effects due to the magnetic field, which one may expect at high pressures,† may not occur, since the current redistribution is determined not only by processes that take place at the wall. Among the determining factors, is, in particular, the magnetic-field dependence of the discharge parameters.

In conclusion I consider it my pleasant duty to thank Professor Ia. P. Terletskii and Lecturer A. A. Zaitsev for their valuable advice and comments in performing the work, as well as to Professors G. V. Spivak and E. M. Reikhrudel' for discussing results. I express my gratitude also to Professor A. Engel for acquainting me with his work.

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*It is impossible to speak of a quantitative comparison in the present case, since the calculation was performed for argon.

†The author is familiar with no experiments in which the influence of the magnetic field vanishes in this way at sufficiently high pressures.

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CLASSIFICATION OF NUCLEAR MAGNETIC MOMENTS

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It is proposed that the region of maximum concentration of the nuclear magnetic moment values be delimited by introducing, along with the Schmidt and Dirac lines, new lines as boundaries of these regions. Empirical data are presented as a proof of the real significance of these lines. It is noted that the quenching of the anomalous part of the nucleon magnetic moment in nuclear matter, which takes place in addition to that from deformation of the nucleus, is, apparently, more important for the formation of nuclear magnetic moments than usually considered.

THE values of magnetic moments are usually divided into two groups—those near the upper and lower Schmidt lines.[1] Together with these, Dirac lines have been proposed.[2] A more detailed classification of the values of μ is both possible and necessary, as follows from the following considerations.

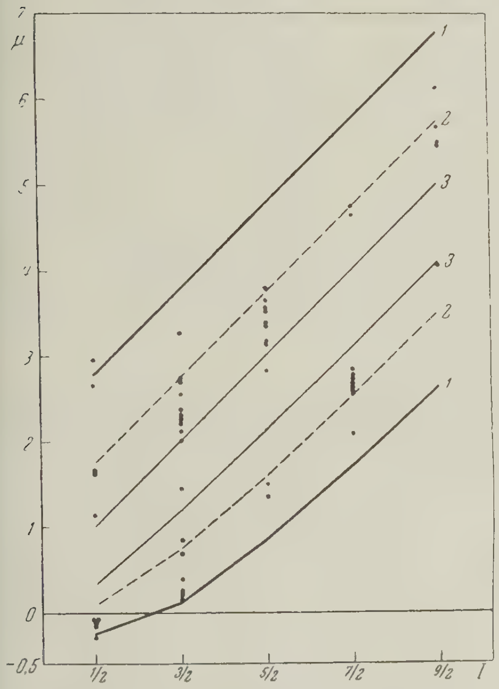

FIG. 1. Classification of magnetic moments of nuclei with odd Z and A. (1) Schmidt line, (2) line a ($\Delta g_s/2 = 1.04$ n.m.), (3) Dirac line.

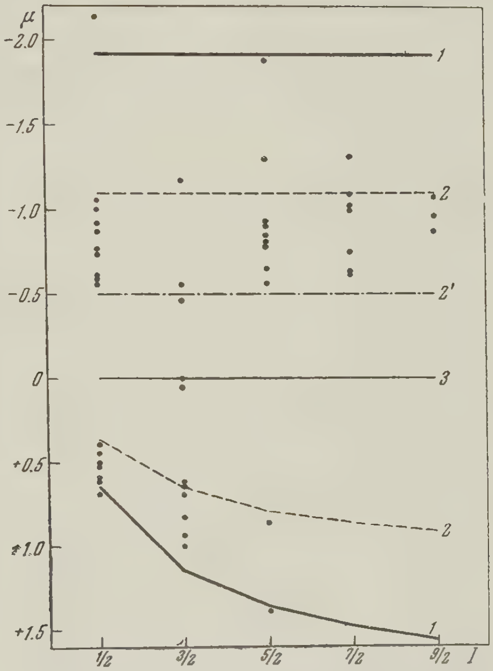

FIG. 2. Classification of magnetic moments of nuclei with odd N and A. (1) Schmidt line, (2) line a ($\Delta g_s/2 = 0.8$ n.m.), (2') line b ($\Delta g_s/2 = 1.4$ n.m.) (3) Dirac line.

1. For $I = \ell + \frac{1}{2}$ (for both odd Z and for odd N) the overwhelming majority of values of μ, as is evident from the table and Figs. 1 and 2, are concentrated in a comparatively narrow strip, running parallel to the Schmidt line but remaining a considerable distance from it. Only a few values of μ lie between this strip and the Schmidt line. This is one of the reasons to separate, for $I = \ell + \frac{1}{2}$, a group of values constituting this strip, on the one hand, and a group of values lying between this strip and the upper Schmidt line, on the other. We designate the line Aa as the boundary between these groups. The Dirac line is the other limit of the strip of maximum concentration of values of μ for $I = \ell + \frac{1}{2}$ for nuclei with odd Z. For nuclei with odd N and with $I = \ell + \frac{1}{2}$, one should draw a special line, which, as is shown by values of μ for spins $\frac{1}{2}$, $\frac{3}{2}$ and $\frac{5}{2}$, should pass at a height of approximately -0.5 nuclear magnetons (n.m.), parallel to the Schmidt line, at a distance of approximately 1.4 n.m. from it. We designate it as the line Ab. In connection with this it is possible for $I = \ell + \frac{1}{2}$ to distinguish in the area between the Schmidt line and Dirac line (for odd Z—the lower Dirac

TABLE. Deviations of the value of the nuclear magnetic moment from the Schmidt line, calculated as changes in the $g_s/2$-factor (in nuclear magnetons)

Odd Z

	Group	Line	1/2 Z	1/2 $\Delta g_s/2$	3/2 Z	3/2 $\Delta g_s/2$	5/2 Z	5/2 $\Delta g_s/2$	7/2 Z	7/2 $\Delta g_s/2$	9/2 Z	9/2 $\Delta g_s/2$
$I = l + 1/2$	A-1-Z	Line Aa ($\Delta g_{s/2}$=1.04)	1	−0.19								
			9	0.16								
					3	0.53						
									23	0.64	41	0.63
	A-2-Z		81	1.16	37	1.04	59	0.99	21	1.03	43	1.11
			81	1.18	5	1.10	13	1.15	27	1.14		
					31	1.23					49	1.29
							55	1.31			49	1.30
							25	1.32				
							63	1.39				
					29	1.41	51	1.43				
					35	1.52						
					29	1.56						
					11	1.57	75	1.61				
			15	1.66	35	1.68	75	1.65				
		Upper Dirac Line ($\Delta g_{s/2}$ =1.79)			31	1.77						
	A-3-Z						53	1.98				
		Lower Dirac Line ($\Delta g_{s/2}$=1.79)			33	2.35						
$I = l - 1/2$	B-2-Z	Line Ba ($\Delta g_{s/2}$=1.04)									83	1.78
									55	1.44		
									57	1.35		
									55	1.30		
									53	1.16		
					17	1.16			71	1.13		
									55	1.11		
									51	1.05		
	B-1-Z				17	0.92						
							63	0.89				
							37	0.69				
			45	0.46	19	0.44			72	0.49		
			47	0.46								
			47	0.40								
			39	0.37	19	0.14						
					77	0.12						
			7	−0.05	79	0.02						

Odd N

1/2 N	1/2 $\Delta g_s/2$	3/2 N	3/2 $\Delta g_s/2$	5/2 N	5/2 $\Delta g_s/2$	7/2 N	7/2 $\Delta g_s/2$	9/2 N	9/2 $\Delta g_s/2$	Line	Group	
1	−0.22									Line Aa ($\Delta g_{s/2}$= 0.8)	A-1-N	$I = l + 1/2$
				9	0.02							
				51	0.61	23	0.60					
		5	0.73									
69	0.86					27	0.81	49	0.82		A-2-N	
67	0.91			55	0.98	45	0.90					
65	0.99			53	1.00	83	0.91	47	0.94			
73	1.03			13	1.06			41	1.03			
75	1.14			143	1.11							
71	1.18			25	1.12	85	1.15					
65	1.29					87	1.27					
63	1.32			103	1.26	85	1.29					
15	1.36	121	1.35	59	1.34							
		29	1.44							Line Ab, ($\Delta g_{s/2}$=1.4)	A-3-N	
		33	1.91							Dirac Line ($\Delta g_{s/2}$=1.91)		
		31	1.96									
											B 2-N	$I = l - 1/2$
		17	0.84							Line Ba ($\Delta g_{s/2}$=0.8)	B-1-N	
		113	0.83									
		77	0.76									
145	0.7			37	0.69							
101	0.56	79	0.53									
119	0.41											
43	0.32	81	0.34									
		19	0.24									
125	0.14											
117	0.08											
7	−0.19			147	−0.07							

line) three groups of values of μ, which we call A-1-Z, A-2-Z, A-3-Z, A-1-N, A-2-N, A-3-N, where A serves as a designation of parallel orientation of the spin and orbital moments. Here the group A-3-Z coincides with the "forbidden zone" between upper and lower Dirac lines.

The moments for the nuclei with Z = 33 and 53 lie in the "forbidden zone." The magnitude of the magnetic moment of the nucleus As^{75} with Z = 33 is nearer to the lower Schmidt line than to the upper. However, with respect to its spin ($3/2$) this nucleus does not differ from the series of nuclei preceding and following it, whose spins and orbital moments are parallel. Also, from the point of view of the shell model, there is no basis to propose the state $d_{3/2}$ in this case. The nucleus I^{127} with Z = 53 has the same spin $5/2$ as the nucleus Sb^{121} with Z = 51 and $I = \ell + 1/2$. Therefore the magnetic moments of the "forbidden zone" can be attributed to group A.

For $I = \ell - 1/2$ the deviations of values of μ from the Schmidt line are, on the average, considerably smaller than for $I = \ell + 1/2$, and the majority of these values are concentrated in a strip bounded, on one side, by the Schmidt line, and on the other by the line Ba, which is analogous in nature to the line Aa (B designates an antiparallel orientation of spin and orbital moments). For $I = \ell - 1/2$ the groups B-1-Z and B-1-N can be distinguished between the Schmidt line and the line Ba, and the groups B-2-Z and B-2-N within the line Ba and the Dirac line (for odd Z — the lower Dirac line).

2. From Figs. 1 and 2 and the table it is clear that for $I = \ell + 1/2$ the values of μ which form the upper limit of each of the groups A-2, almost all deviate from the Schmidt line. For odd Z and $I = \ell + 1/2$ they show precisely the same dependence on nuclear spin as the upper Schmidt line, forming the series 1.6, 2.8, 3.8, 4.8, 5.7 n.m. For odd N the difference between the Schmidt numbers and the values of μ which form the upper boundary of the group A-2 remains almost constant with change in I. Since this difference for both odd Z and odd N does not exhibit any appreciable dependence on the magnitude of ℓ and I, it is possible that it comes from the change of only the spin part of the magnetic moment $g_s/2$, on account of its deviation from the magnitude of the magnetic moment of a free nucleon.

Thus, for the value of μ forming the upper limit of each of the groups A-2, we can take, to a good approximation, the formula

$$\mu = g_l l + g_{s1} s, \tag{1}$$

representing a Schmidt formula, with the part relating to the spin moment changed. Here it is possible to take for protons $g_\ell = 1$, $g_{s1}/2 = g_s/2 - 1.04$ n.m. and for neutrons $g_\ell = 0$, $g_{s1}/2 = g_s/2 + 0.8$ n.m. There is an analogous line for $I = \ell - 1/2$. Its formula is

$$\mu = g_l (l+1)(2l-1)/(2l+1) - g_{s1}(2l-1)/2(2l+1). \tag{2}$$

The line Ba, representing this formula, for odd N coincides sufficiently closely with the values of μ which deviate most from the Schmidt line. It is important that the value of $\Delta g_s/2$ for the lines Aa and Ba is the same. For odd N one thus observes the coincidence of maximum deviation of $g_s/2$ for $I = \ell - 1/2$ with the minimum of mass deviations for $I = \ell + 1/2$. For odd Z the line Ba for $I = 1/2, 3/2, 5/2$ is near to the maximum limit of the deviations, and for $I = 7/2$ it coincides accurately with their minimum limit. The lines Aa and Ba for odd Z are somewhat further from the Schmidt line than for odd N, as is to be expected.

3. The nuclei whose moments enter into groups A-1-Z and A-1-N are characterized by several particularities in their shell structure; almost all of them have only one nucleon above a closed orbital term* (or above a doubly-magic core), whereas all relate to the first half of the periodic system where, following the closing of the orbital term, a much greater change occurs in the energy level than in the second half [after closing a sub-orbital term the deviation from the Schmidt line is almost always very large, that is, after (a) 14, 28 or 82 protons or neutrons; (b) 50 protons; (c) 126 neutrons, especially in the Bi^{209} nucleus, after the simultaneous closing of two sub-orbital terms]. It is interesting that of nine values of μ relating to group A-1, the deviations of the four which differ most from the Schmidt lines are, for both odd Z and odd N, practically the same and near to 0.6 n.m.

4. Mayer and Jensen,[3] in considering the appearance of isomers, indicate that the magnitude of the matrix element for the M-4 transition can be brought into agreement with experiment if, in the relevant equations, the value 0.8 n.m. is substituted for μ_p rather than its value for a free nucleon, and the value

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*By orbital term we understand the totality of adjacent nucleons of the same type having the same ℓ; by sub-orbital, having the same j.

0.6 n.m. for μ_N. The magnitudes of these are near, for protons, to one of the Dirac lines, and for neutrons, to one of the lines proposed by us (Ab).

We have justification to expect that in the future other phenomena will be found confirming the real significance of both the Dirac line and the line proposed by us.

It would appear that the simplest and most natural explanation of the characteristic properties of the lines proposed by us could be given by the hypothesis[2,4,5] about the quenching of the anomalous part of the magnetic moment of the unpaired nucleon in nuclear matter. (We note that acceptance of the theory of quenching means acceptance of the single-particle model only as applied to the spin, and not to the orbital part of the nuclear magnetic moment.) In several cases the influence of the quenching appears very clearly, for example, in relation to μ of Bi^{209}, the value of which (4.08 n.m.) almost exactly coincides with the Dirac line (4.09 n.m.). We add that the value of μ for Bi^{209} cannot be explained either from the theory of $j-j$ coupling between proton and neutrons,[6] according to which there should be no deviation from the Schmidt line in the given case, nor by the collective model,[7] in view of the small value of the quadrupole moment of this nucleus. The calculations of Blin-Stoyle[8] give only a very ill-defined value for the deviation, 0.81—1.68 n.m., and the calculations of Arima and Horie[9] a value of 0.81 n.m., with an overall deviation of 1.46 n.m.

Together with the coincidence noted above of the deviations for four values of μ which are part of the groups A-1-Z and A-1-N, it is impossible to believe such facts to be purely coincidental as, for example, the exact agreement of the deviations of the pair of nuclei $_{37}Rb_{48}$ and $_{30}Zn_{37}$ if a change in the g_S-factor is taken as the deviation. It is characteristic that just these two nuclei differ from all preceding odd nuclei also by other particularities: a spin $5/2$ in place of $3/2$ for $I = \ell - 1/2$ in place of $I = \ell + 1/2$. Together with this, in the majority of other cases it is necessary to combine the theories of quenching and deformation. Deformation has a particularly great influence on the values of μ in group A-3. However, the influence of quenching apparently is none the less more important than many authors have assumed (see Ref. 10 and others) and it is possible that their ideas concerning this question should be considered only as evidence of the incompleteness of contemporary concepts about nuclear properties. This would signify that the theory of quenching needs further development on a substantially broader basis.

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RADIATION EMITTED BY MOLECULES IN THE PRESENCE OF A STRONG HIGH-FREQUENCY FIELD

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A strong electromagnetic field of frequency ω and amplitude F induces radiation of lower frequency Ω_0 [Eq. (12)] in a molecular gas. The intensity of this radiation is computed. Quantum-electrodynamical considerations are presented which corroborate the applicability of the correspondence principle in the investigation of this phenomenon.

1. INTRODUCTION

THE phenomenon of radiation by molecules in the presence of a strong high-frequency field has been examined by the author.[1] In the present paper we develop a more detailed analysis of this phenomenon.

Let E_1 and E_2 be two non-degenerate energy levels of a molecule, μ_{11} and μ_{22} the dipole moments (electric or magnetic) of the molecule in the states with energies E_1 and E_2, and μ_{12} the dipole moment of the transition 1 → 2. Thus E_1 and E_2 can be components of the Stark splitting of the levels, whence μ_{11} and μ_{22} are electric dipole moments. Furthermore, let a strong electromagnetic field act at a frequency ω, close to the resonant frequency of the molecule $\omega_{12} = (E_1 - E_2)/\hbar \equiv \omega_0$. As is known, such a field excites transitions between the states 1 and 2 of the molecule. As a result of such transitions a change of the dipole moment of a molecule takes place from μ_{11} to μ_{22} and conversely. This change takes place with the frequency of transitions between the states 1 and 2. The number of such transitions per unit time has the order of magnitude

$$\Omega_0 \approx |\mu_{12}| F / \hbar, \tag{1}$$

where F is the amplitude of the field of frequency ω. Every change of dipole moment, however, must give rise to emission of energy. It is evident that such emission must have the frequency Ω_0. The experimental determination of this frequency permits very accurate evaluation of the product of the dipole moment of the transition μ_{12} by the amplitude of the high-frequency field.

By such a method, in principle, with a known dipole moment, a precise measurement of high-frequency fields is possible. On the other hand, if the intensity of the high-frequency field is known, there is the possibility of direct determination of the dipole moment μ_{12}.

We note that the effect described here is to a certain extent analogous to the Stark and Zeeman effects. In fact, the Stark effect gives the possibility of observing radiation of a frequency proportional to the product of the intensity of the constant electric field by the electric dipole moment in a given energy state (in our case μ_{11} or μ_{22}); the Zeeman effect gives the possibility of observing a frequency proportional to the product of the intensity of the constant magnetic field by the magnetic dipole moment (diagonal element of the magnetic dipole-moment operator). In the presence of a strong high-frequency field, however, there appears a radiation of frequency Ω_0, proportional to the product of the amplitude of this field by the magnitude of the dipole moment μ_{12} of the transition.

We shall first calculate the effect described here with the aid of the correspondence principle and then indicate quantum-electrodynamic considerations that confirm the existence of this effect.

2. ANALYSIS BASED ON THE CORRESPONDENCE PRINCIPLE

Based on the correspondence principle, we shall describe a molecule by means of quantum mechanics and the radiation by means of classical electrodynamics.

We shall assume that the following conditions are fulfilled:[2-4]

(a) $T_0 \ll 1/\omega \ll \bar{\tau}$, where T_0 is the duration of a molecular collision and $\bar{\tau}$ is the mean time between collisions;

(b) $$|\omega - \omega_0| \equiv |\omega - \omega_{12}| \lesssim 1/\bar{\tau} \text{ and } |\omega - \omega_{mn}| \gg 1/\bar{\tau},$$

where m, n ≠ 1, 2;

(c) A molecular collision returns a molecule to the state with energy E_1 or E_2;

(d) $\omega_0 \gg \Omega_0$.

Let an isolated molecule have levels E_i which are characterized by the eigenfunctions $\Psi_i \exp(iE_i t/\hbar)$. Then

$$H_0 \Psi_i = E_i \Psi_i, \tag{2}$$

where H_0 is the Hamiltonian of a free molecule. In the presence of interaction with an electromagnetic field of frequency ω, the Hamiltonian of a molecule in an external field will have the form

$$H = H_0 - \mu F(t) = H_0 + V \sin \omega t, \tag{3}$$

where $\mathbf{F}(t) = \mathbf{F} \sin \omega t$ and $\boldsymbol{\mu}$ is the dipole moment operator.

The wave function of a molecule in an external field satisfies the Schrödinger equation

$$i\hbar\, \partial\Psi/\partial t = (H_0 + V \sin \omega t)\, \Psi. \tag{4}$$

We shall seek the solution of this equation satisfying the conditions (a) through (d) by a method given in the book by Landau and Lifshitz.^{5}

In the presence of interaction, we seek a solution of Eq. (4) in the form

$$\Psi(t) = a_1 \Psi_1 + a_2 \Psi_2. \tag{5}$$

Here we have used condition (b). At the instant of collision, $t = t_0$, according to condition (c), let*

$$a_1 = 0; \quad a_2 = 1. \tag{6}$$

Using conditions (b) and (d), we obtain upon transformation the following equations for a_1 and a_2:

$$i\hbar \frac{da_1}{dt} = -\frac{V_{12}}{2i} e^{i\delta t} a_2; \; i\hbar \frac{da_2}{dt} = \frac{1}{2i} V_{21} e^{-i\delta t} a_1, \tag{7}$$

where $\delta = \omega_0 - \omega$. We make the substitutions

$$D = |a_2|^2 - |a_1|^2; \quad c_{12} = a_1^* a_2; \quad c_{21} = a_2^* a_1 \tag{8}$$

and obtain the equations

$$\dot{D} = -[V_{12} c_{21} e^{i\delta t} + V_{21} c_{12} e^{-i\delta t}]/\hbar, \tag{9a}$$
$$\dot{c}_{12} = (V_{12}/2\hbar) D e^{i\delta t}, \; \dot{c}_{21} = (V_{21}/2\hbar) D e^{-i\delta t}. \tag{9b}$$

Equations (9) can be solved exactly. Differentiation of Eq. (9a) and substitution in Eq. (9b) gives

$$\ddot{D} = -\hbar^{-2} |V_{12}|^2 D - (i\delta/\hbar) [V_{12} c_{21} e^{i\delta t} - V_{21} c_{12} e^{-i\delta t}].$$

Again differentiating and using Eqs. (9a, b) we obtain

$$D = -\{(|V_{12}|/\hbar)^2 + \delta^2\} \dot{D}. \tag{10}$$

The general integral of this equation is

$$D = \alpha \cos \Omega_0 (t - T) + \beta, \tag{11}$$

where

$$\Omega_0^2 = (|V_{12}|/\hbar)^2 + \delta^2 \tag{12}$$

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*One could choose the other possible initial condition $a_2 = 0$, $a_1 = 1$. The final result, as is clear from physical considerations, does not depend on which of these conditions is assumed.

and α, β and T are real constants. The initial conditions [Eq. (6)] can be rewritten in the new notation in the form

$$c_{12} = c_{21} = 0, \quad D = 1 \ \text{for} \ t = t_0. \tag{13}$$

If the first two conditions are substituted in Eq. (9a) we obtain $\dot{D} = 0$ when $t = t_0$; whence $T = t_0$ in Eq. (11). Analogously we find the constants α and β. As a result we obtain

$$D = (\hbar\Omega_0)^{-2} \{ |V_{12}|^2 \cos \Omega (t - t_0) + (\hbar\delta)^2 \}. \tag{14}$$

Let us now determine the mean dipole moment of a molecule at an instant of time t. Using Eq. (5), we find

$$\langle \mu \rangle = \int \Psi^* (t)\, \mu\, \Psi (t)\, dq = |a_1|^2 \mu_{11} + |a_2|^2 \mu_{22} + c_{12} \mu_{12} e^{-i\omega_0 t} + c_{21} e^{i\omega_0 t} \mu_{21}. \tag{15}$$

Using the corresponding principle as a basis, we can now find the emission and absorption of the molecule. The last two terms in Eq. (15) give the contribution to absorption of electromagnetic radiation of frequency ω.[2,3] Therefore, we shall be interested in the first two terms in Eq. (15):

$$\langle \mu_1 \rangle = |a_1|^2 \mu_{11} + |a_2|^2 \mu_{22} = {}^1/_2 [\mu_{11} + \mu_{22} + (\mu_{22} - \mu_{11}) \Omega_0^{-2} \delta^2] + {}^1/_2 (\hbar\Omega_0)^{-2} (\mu_{22} - \mu_{11}) |V_{12}|^2 \cos \Omega_0 (t - t_0). \tag{16}$$

Here we have used Eq. (14) and the fact that $|a_1|^2 + |a_2|^2 = 1$. The contribution to the emission is given by the variable part of the moment

$$\langle \tilde{\mu} \rangle = \mu_0 \cos \Omega_0 (t - t_0), \tag{17}$$

where*

$$\mu_0 = {}^1/_2 (\mu_{22} - \mu_{11}) (\hbar\Omega_0)^{-2} |V_{12}|, \tag{18}$$

in the case of strict resonance $\delta = 0$ and

$$\mu_0 = {}^1/_2 (\mu_{22} - \mu_{11}); \quad \Omega_0 = |V_{12}| / \hbar. \tag{19}$$

Taking into account that the times τ between collisions are distributed according to the law

$$w(\tau)\, d\tau = (1/\bar{\tau}) \exp (-\tau/\bar{\tau})\, d\tau \tag{20}$$

(this occurs for a sufficiently rarefied gas), one can find that the spectral intensity of the emission of a molecule has the form

$$I(\Omega) \sim [(\Omega_0 - \Omega)^2 + \bar{\tau}^{-2}]^{-1}. \tag{21}$$

The total intensity of the emission is equal to

$$I_{\text{total}} = (\Omega_0^4 / 3c^3)\, \mu_0^2. \tag{22}$$

This emission is essentially a non-equilibrium process. It proceeds only under the influence of an external driving field (in our case the strong field of frequency ω). Therefore Kirchhoff's law does not apply here and the absorption by a molecule of a field of frequency Ω_0 will not bear the resonance character of Eq. (21).

In fact, in addition to the strong field of frequency ω, let a weak field of frequency Ω close to the frequency Ω_0 act on a molecule. Then the Hamiltonian of a molecule in an external field can be written

$$H = H_0 + V \sin \omega t + W \sin \Omega t. \tag{23}$$

In this case in place of Eqs. (9) we obtain

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*In Ref. 1 in the expression for μ_0 there is an additional factor $D_0 = \rho_{22}^0 - \rho_{11}^0$ or, in the notation of the present work, $D_0 = (|a_2|^2 - |a_1|^2)_{t = t_0}$. This quantity, however, has a modulus of unity (see the preceeding footnote) and we therefore omit it here.

$$\dot{D} = -\frac{1}{\hbar}[V_{12}c_{21}e^{i\delta t} + V_{21}c_{12}e^{-i\delta t}] + \frac{2\sin\Omega t}{i\hbar}[W_{21}e^{i\omega_{21}t}c_{12} - c_{21}W_{12}e^{i\omega_{12}t}];$$

$$\dot{c}_{12} = \frac{1}{2\hbar}V_{12}De^{i\delta t} + \frac{1}{i\hbar}\sin\Omega t\,[W_{12}e^{i\omega_{12}t} + c_{12}(W_{11} - W_{22})]; \quad \dot{c}_{21} = \frac{1}{2\hbar}V_{21}De^{i\delta t} - \frac{\sin\Omega t}{i\hbar}[W_{21}e^{i\omega_{21}t} + c_{21}(W_{11} - W_{22})].$$

If we discard the non-resonance terms of these equations we obtain

$$\dot{D} = -\frac{1}{\hbar}[V_{12}c_{21}e^{i\delta t} + V_{21}c_{12}e^{-i\delta t}]/\hbar; \tag{24a}$$

$$\dot{c}_{12} = (V_{12}/2\hbar)De^{i\delta t} + \frac{\sin\Omega t}{i\hbar}(W_{11} - W_{22})c_{12}, \quad \dot{c}_{21} = (V_{21}/2\hbar)De^{-i\delta t} - \frac{\sin\Omega t}{i\hbar}(W_{11} - W_{22})c_{21}. \tag{24b}$$

Examining, solely for simplicity, the case of strict resonance $\delta = 0$, differentiation of Eq. (24a) yields

$$\ddot{D} + \hbar^{-2}|V_{12}|^2 D = \frac{1}{i\hbar}\sin\Omega t\,(W_{11} - W_{22})[V_{12}c_{21} - V_{21}c_{12}]. \tag{25}$$

In the absence of a field of frequency Ω, the right side of Eq. (25) is zero. If absorption of the frequency Ω exists, it is caused by the term on the right side of Eq. (25). It is easy to see, however, that in our approximation the right side is zero. As a matter of fact, one can substitute Eqs. (9b) for c_{12} and c_{21} in the right side. But from these equations we have (for $\delta = 0$):

$$(d/dt)[V_{12}c_{21} - V_{21}c_{12}] = 0.$$

At the initial moment $t = t_0$; $c_{12} = c_{21} = 0$ and consequently for all time

$$V_{12}c_{21} - V_{21}c_{12} = 0.$$

Thus a weak external field of frequency $\Omega \approx \Omega_0$ does not change the character of the motion of a molecule and consequently, resonance absorption at the frequency Ω_0 does not exist. This also follows from the physical considerations presented above.

Let us now examine the question of the coherence of the radiation from the individual molecules of a gas. Let for example, E_1 and E_2 be components of the Stark splitting and μ_{11} and μ_{22} be effective dipole moments in the states 1 and 2. (In this case it is necessary to assume that the effect of a constant electric field is taken into account in the Hamiltonian of the "free" molecule H_0.) Then μ_0 [see Eq. (18)] will have one and the same value for all molecules and consequently, the amplitudes of the radiation fields of all molecules will have one and the same value.* The phase of the radiation however, is random and depends on the time of the last collision for each molecule. The total field intensity of the radiation of all the molecules will be:[6]

$$\mathbf{E} = \sum_i \mathbf{A}_i \cos\Omega(t - t_{i0}) = \mathbf{A}\sum_i \cos(\Omega t - \varphi_i) = \mathbf{E}_0\cos(\Omega t + \varphi), \tag{26}$$

$$\mathbf{E}_0^2 = A^2\Big[\Big(\sum_i \cos\varphi_i\Big)^2 + \Big(\sum_i \sin\varphi_i\Big)^2\Big]; \tag{27}$$

$$\tan\varphi = \sum_i \sin\varphi_i \Big/ \sum_i \cos\varphi_i. \tag{28}$$

Equation (27) can be rewritten in the form

$$E_0^2 = A^2\Big[n + 2\sum_{i,k}\cos(\varphi_i - \varphi_k)\Big], \tag{29}$$

where n is the number of molecules and the summation is carried out over the cosines of the $n(n-1)/2$ phase differences between the separate molecules.

Let us estimate the possible values of the phase differences $\epsilon_{ik} = \varphi_i - \varphi_k$. For all $\epsilon_{ik} \ll 1$, $E_0^2 \approx n^2A^2$ and the radiation is coherent; for ϵ_{ik}, which take all possible values from 0 to 2π and higher, the sum in Eq. (29) is equal to zero and the radiation is incoherent. In our case

$$\varepsilon_{ik} = \Omega(t_{i0} - t_{k0}).$$

For an estimate one can assume that $t_{10} - t_{k0}$ varies from 0 to $\bar{\tau}$—the mean time between collisions. But it is obvious that the condition $\Omega\bar{\tau} > 2\pi$ is essential for the existence of the spectral line Ω_0. Thus we can rewrite Eq. (26) in the form:

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*In the example under discussion it is supposed that the molecules are separated by distances much smaller than the wavelength of the emitted radiation.

$$\mathrm{E} = \mathrm{A}\sqrt{n}\cos(\Omega t + \varphi),$$

and the intensity of radiation of all the molecules is obtained as the product of the intensity of radiation of a single molecule [Eq. (22)] by the number of molecules n:

$$J_s = (\Omega_0^4/3c^3)\mu_0^2 n. \tag{30}$$

Reabsorption of radiation at the frequency Ω_0 can be disregarded because there is no resonance absorption at the frequency Ω_0. We note that, in contrast to the intensity of spontaneous emission, which is proportional to the number of active molecules $n_{\mathrm{act}} = n_2 - n_1$ (see, e.g., Refs. 7 – 9), where n_1 and n_2 are the numbers of molecules in the upper and lower levels, it is the total number of molecules which enters in Eq. (30). This is related to the circumstance that the intensity of radiation at the frequency Ω_0 does not depend on whether a molecule is found in the upper or lower level at the instant of collision.

The effect described here of radiation in the presence of a strong high-frequency field occurs only in a case where the difference $(\mu_{22} - \mu_{11})$ is not zero. Generally speaking, a check of this condition is not necessary in each specific case. One can say at once, however, that the indicated difference does not vanish for the case where $E_1 - E_2 = \hbar\omega_0$ depends on a constant electric (Stark effect) or magnetic (Zeeman effect) field.

As a matter of fact in this case the energy of interaction with the constant field F_c is included in the Hamiltonian of the "free" molecule

$$H_0' = H_0 - \mu_z F_c$$

(the steady field F_c is directed along the z axis). Further, since

$$(\partial H_0'/\partial F_c)_{nn} = -(\mu_z)_{nn} = \partial E_n/\partial F_c$$

(see, e.g., Ref. 5), we have

$$(\mu_z)_{22} - (\mu_z)_{11} = -\partial(E_2 - E_1)/\partial F_c.$$

Let us introduce the appropriate matrix elements for diatomic molecules and symmetric-top type molecules.

We examine first the diatomic molecule. If the molecule consists of different atoms (or of different isotopes), it has a constant dipole moment $\mu\mathbf{n}$, where $\mathbf{n}$ is a unit vector directed along the axis of the molecule. It will be necessary for us to determine the matrix elements $\mu_z = \mu n_z$.

In Hund's case a (see Ref. 5)

$$E(J, M_J, \Omega) = U_e + A_e\Omega + \hbar\omega_e(v + 1/2) + B_e[J(J+1) - 2\Omega^2] - F_c\mu M_J\Omega/J(J+1).$$

Here E is the energy level of a molecule in the presence of an electric field F_c; $\Omega = \Lambda + \Sigma$ is the projection of the sum of the orbital and spin moments of a molecule on the axis of the molecule, J is the total moment of the number of motions of the molecule, and M_J is its projection on the z axis. If a high-frequency field F is directed along the z axis, the selection rules $\Delta J = \pm 1$; and $\Delta M_J = 0$ apply.

The difference of the energy levels is

$$E(J+1, M_J, \Omega) - E(J, M_J, \Omega) = 2B_e(J+1) - F_c\mu\frac{2M_J\Omega}{J(J+1)(J+2)};$$

whence

$$(\mu_z)_{22} - (\mu_z)_{11} = 2\mu M_J\Omega/J(J+1)(J+2).$$

We note that in the absence of an electric field there is degeneracy over the quantum numbers M_J which is completely removed by the linear Stark effect. Thus in our case the presence of an electric field is necessary since in the absence of the field the mean value of $(\mu_z)_{22} - (\mu_z)_{11}$ over all the degenerate sublevels of M_J is zero.

The matrix elements $(\mu_Z)_{12}$ in the present case have the form[5]

$$(\mu_z)^{J+1,\,M_J,\,\Omega}_{J,\,M_J,\,\Omega} = \mu \sqrt{(J+1)^2 - M_J^2}\,\frac{1}{J+1}\sqrt{\frac{(J+1)^2-\Omega^2}{4\,(J+1)^2-1}}\,.$$

In an analogous way one can find matrix elements for Hund's case b. In this case the energy of the Stark splitting is

$$-F_c\,(\mu_z)_{nn} = \Delta E_{M_J} = -F_c \mu M_J \Lambda\,\frac{J\,(J+1) - S\,(S+1) + K\,(K+1)}{2K\,(K+1)\,J\,(J+1)}\,.$$

Here Λ is the projection of the orbital moment of the electrons on the axis of the molecule, S is the total spin of all the electrons of the molecule, and K is the sum of the orbital moment of the electrons and of the rotational moment of the nuclei.

For non-linear molecules of the symmetric-top type it is easy to find[5]

$$(\mu_z)_{22} - (\mu_z)_{11} = 2\mu M_J k / J\,(J+1)\,(J+2);\quad (\mu_z)^{J+1,\,M_J,k}_{J,\,M_J,\,k} = \mu\sqrt{(J+1)^2 - M_J^2}\,\frac{1}{J+1}\sqrt{\frac{(J+1)^2 - k^2}{4\,(J+1)^2 - 1}}\,,$$

where J is the moment of the top, M_J its projection on the z axis and k the projection on the moving axis connected to the molecule. One should note that the matrix element $(\mu_Z)_{12}$ which determines the transition probability between the levels E_1 and E_2, generally speaking is different from the matrix element $(\mu_Z^0)_{12}$ which corresponds to the transition between the levels E_1^0 and E_2^0, where the index 0 denotes levels of the Hamiltonian H_0 (without a constant field). In case, however, the energy of the Stark splitting is much less than the difference of the levels of the non-perturbed Hamiltonian (only under this condition are all the formulas written down here correct), the difference of these elements is rather small.

With the aid of the formulas written down here for the matrix elements one can estimate the value of the intensity of the radiation Eq. (30). Let $\delta = 0$, then, according to Eq. (19)

$$\Omega_0 = F\,(\mu_z)_{12}/\hbar;\quad \mu_0 = {}^1\!/_2\,[(\mu_z)_{22} - (\mu_z)_{11}]$$

and for the symmetric-top we have*

$$J_s = \frac{F^2\mu^6}{3c^3\hbar^4}\,n\,\frac{[(J+1)^2 - M_J^2\,]^2[(J+1)^2 - k^2\,]^2 M_J^2 k^2}{J^2\,(J+1)^6\,(J+2)^2\,[4\,(J+1)^2 - 1]^2}\,.$$

Let $k = 1$, $M_J = 1$ and $J = 1$, then

$$J_s = 7.7\cdot 10^{-43}\,n\mu^6 F^4\ \mathrm{w}\,.$$

Here μ is in Debye units and F is cgs units. If we take $\mu = 12$ (for the CsI molecule, see Ref. 10), $n = 10^{17}$ and $F = 5$ statvolts/cm = 1.5 kv/cm, then $J_s = 1.4 \times 10^{-16}$ w.

The intensity of radiation can be substantially raised by means of a molecular beam in which all of the molecules are in a single energy state. If a high-frequency field begins to act on a molecule in such a molecular beam, then in the first instant the system will radiate coherently with an intensity proportional to n^2, the square of the number of molecules in the beam. The dipole moment of the entire system will be equal to $n\mu_0 \cos\Omega_0\,(t - t_0)$. Newly arriving molecules, however, will decrease the dipole moment of the system since the sum of the dipole moments of these molecules is close to zero.

Actually, let the field begin to act at the time t_0. Then the dipole moment of the entire system at the time t is equal to

$$\mathbf{P} = n_1\,(t)\,\boldsymbol{\mu}_0 \cos\Omega_0\,(t - t_0) + \sum_{i=1}^{n_2\,(t)} \boldsymbol{\mu}_0 \cos\Omega_0\,(t - t_0 - z_i\,/\,v_i),$$

where $n_2(t)$ is the number of molecules newly arrived in the time $t - t_0$ $[n_2(t_0) = 0]$ and $n_1(t)$ is the number of molecules still remaining of those which in the initial moment filled the entire region in which the field was acting, $[n_1 = n - n_2]$; z_i and v_i are the coordinates and velocity of the i-th molecule.

At the instant of time $t_0 + \tau$, where τ is the mean time of flight of a molecule through the condenser (in which the high-frequency field acts) $n_1\,(t_0 + \tau) = 0$ and the dipole moment

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*The same is true for diatomic molecules in a state with spin equal to zero if one substitutes J, M_J and k for K, M_K and Λ.

$$\mathbf{P} = \sum_{i=1}^{n} \boldsymbol{\mu}_0 \cos \Omega_0 (t - t_0 - z_i / v_i) \approx 0.$$

The radiation becomes completely incoherent. If the field is now switched off, then after a time $\geq \tau$, i.e., at an instant of time $t_0' \geq t_0 + 2\tau$, all of the molecules in the condenser will be newly found in a single energy state. If at this moment the field is again switched on, radiation $\sim n^2$ once more resumes which after a time τ completely goes over into non-coherent radiation $\sim n$.

Thus in order to attain coherent radiation of a frequency Ω_0 it is necessary to supply a high-frequency field in pulses of duration $t_1 \leq \tau$, where the time between pulses t_2 must satisfy the condition $t_2 \geq \tau$. Then in the time interval t_1 the radiation Ω_0 will be approximately proportional to n^2 while in the interval t_2 the intensity will be zero. Thus, for $n = 10^{10}$ (this is a fully attainable number of molecules in a molecular beam; cf. Ref. 11) $n^2 = 10^{20}$, this gives an intensity of radiation $\sim 10^3$ larger than that calculated earlier.

3. QUANTUM-ELECTRODYNAMIC ANALYSIS

The occurrence radiation at the frequency Ω_0 [Eq. (12)] can be explained on the basis of simple quantum-electrodynamic considerations.

Actually a system which consists of a molecule and of an electromagnetic field of frequency ω, is described by the Hamiltonian

$$\mathscr{H} = H_0 + H_\gamma^0 + V/2, \tag{31}$$

where H_0 is the Hamiltonian of the free field of frequency ω and $V/2$ is the interaction energy [the coefficient $^1/_2$ is needed to retain conformity with Eq. (30)]. We shall seek a solution of the equation

$$\mathscr{H}\Psi = \mathscr{E}\Psi \tag{32}$$

in the form of a superposition of solutions of the equation

$$(H_0 + H_\gamma^0)\, \Psi^{(0)} = \mathscr{E}^{(0)}\Psi^{(0)}.$$

Let us write

$$\Psi = c_1 \Psi_1^{(0)} + c_2 \Psi_2^{(0)}, \tag{33}$$

where

$$\Psi_1^{(0)} = \Psi_{1m}^{(0)} \Phi_{N_0}(N); \quad \Psi_2^{(0)} = \Psi_{2m} \Phi_{N_0+1}(N)$$

are wave functions which correspond to energy levels of the system molecule + field:

$$\mathscr{E}_1^{(0)} = E_1 + N_0 \hbar\omega; \quad \mathscr{E}_2^{(0)} = E_2 + (N_0 + 1)\,\hbar\omega,$$

Here $\mathcal{E}_1^{(0)} \approx \mathcal{E}_2^{(0)}$; $\Psi_{1m}^{(0)}$ and $\Psi_{2m}^{(0)}$ are wave functions of the free molecule, $\Phi_{N_0}(N)$ and $\Phi_{N_0+1}(N)$ are wave functions of the free field and N is the number of photons

Substituting Eq. (33) in Eq. (32), multiplying first by $\Psi_1^{(0)*}$ and then by $\Psi_2^{(0)*}$ and integrating, we obtain two equations

$$c_1 (\mathscr{E}_1^{(0)} - \mathscr{E}) + c_2 V_{12}/2 = 0, \quad c_1 V_{21}/2 + c_2 (\mathscr{E}_2^{(0)} - \mathscr{E}) = 0. \tag{34}$$

The condition for the existence of a non-trivial solution to these equations gives

$$\mathscr{E}_{1,2} = \frac{1}{2}(\mathscr{E}_1^{(0)} + \mathscr{E}_2^{(0)}) \pm \frac{1}{2}\sqrt{(\mathscr{E}_1^{(0)} - \mathscr{E}_2^{(0)})^2 + |V_{12}|^2}. \tag{35}$$

Thus, transitions of the molecule + field system are possible with radiation of quanta of energy

$$\mathscr{E}_1 - \mathscr{E}_2 = \sqrt{\hbar^2 (\omega_0 - \omega)^2 + |V_{12}|^2} = \hbar\Omega_0. \tag{36}$$

This is the same expression for the frequency of the emission which we obtained earlier [see Eq. (12)].

On the basis of the correspondence principle we have found that the intensity of radiation is proportional to $|\mu_{11} - \mu_{22}|^2$. A quantum electrodynamic examination must lead to the conclusion that the matrix elements of the operator of the energy of interaction with the electromagnetic field of frequency Ω_0 which correspond to the transition $\mathcal{E}_1 \to \mathcal{E}_2$, are proportional to $|\mu_{11} - \mu_{22}|$. We shall demonstrate that this actually occurs.

The operator for the energy of interaction with the field of the emission can be written as

$$\hat{W} = \hat{W}_0(q) \sum_k{}' (a_k + \overset{+}{a}_k) = B\hat{\mu} \sum_k{}' (a_k + \overset{+}{a}_k); \qquad (37)$$

Here $\hat{W}_0(q)$ depends only on the coordinates of the molecule, B is a constant and a_k and $\overset{+}{a}_k$ are operators of absorption and creation of photons of frequency $\Omega = ck$. The prime on the summation sign denotes that terms with frequency close to ω_0 are dropped from the sum, these terms being taken into account in Eq. (31) in the interaction energy V/2 with the external field of frequency ω. The matrix elements of the operator Eq. (37) have the form:

$$\left(\Psi_{\mathcal{E}_1} \Phi_{N_\Omega^0}(N_\Omega),\ \hat{W}_0(q) \sum_k{}' (a_k + a_k^+) \Psi_{\mathcal{E}_2} \Phi_{N_\Omega^0 + 1}(N_\Omega) \right). \qquad (38)$$

Let further

$$\Psi_{\mathcal{E}_1} = c_1 \Psi_{1m} \Phi_{N_0}(N) + c_2 \Psi_{2m} \Phi_{N_0+1}(N), \quad \Psi_{\mathcal{E}_2} = c_1' \Psi_{1m} \Phi_{N_0}(N) + c_2' \Psi_{2m} \Phi_{N_0+1}(N). \qquad (39)$$

Then the matrix element which corresponds to the appearance of a photon of frequency Ω, is

$$(\mathcal{E}_1, 0 | \hat{W} | \mathcal{E}_2, 1_\lambda) = \overset{*}{c}_1 c_1 ((\hat{W}_0 q))_{11} + \overset{*}{c}_2 c_2' (\hat{W}_0(q))_{22} = B (\overset{*}{c}_1 c_1' \mu_{11} + \overset{*}{c}_2 c_2' \mu_{22}). \qquad (40)$$

Here we have taken into account the orthogonality of the functions Φ_{N_0} and Φ_{N_0+1}. The probability of a transition with the emission of a photon with frequency Ω is proportional to the square of the modulus of the matrix element Eq. (40) and is essentially different from zero under the condition that $\mathcal{E}_1 = \mathcal{E}_2 + \hbar\Omega$. This condition coincides with Eq. (36).

It is still necessary for us to find the coefficient c_1, c_1', c_2 and c_2'. From Eqs. (34) and (35) we find

$$c_1 / c_2 = V_{12} / [\mathcal{E}_2^{(0)} - \mathcal{E}_1^{(0)} + \sqrt{(\mathcal{E}_1^{(0)} - \mathcal{E}_2^{(0)})^2 + |V_{12}|^2}], \quad \overset{*}{c}_1 / \overset{*}{c}_2 = -c_2' / c_1' \quad \text{or} \quad \overset{*}{c}_1 c_1' = -\overset{*}{c}_2 c_2'. \qquad (41)$$

Using Eqs. (40) and (41) we find

$$(\mathcal{E}_1 | \hat{W} | \mathcal{E}_2 + \hbar\Omega) = B \overset{*}{c}_1 c_1' (\mu_{11} - \mu_{22}). \qquad (42)$$

Here under $\boldsymbol{\mu}_{11}$ and $\boldsymbol{\mu}_{22}$ it is necessary to understand the projection of the vectors $\boldsymbol{\mu}_{11}$ and $\boldsymbol{\mu}_{22}$ in the direction of polarization of the emitted quantum $\hbar\Omega$.

Thus the quantum electrodynamic approach justifies the application of the correspondence principle in our case.

In conclusion I should like to express deep thanks to Professor V. L. Ginzburg for his valuable advice and thorough discussion of the results of the work.

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ON SOME ELECTROMAGNETIC EFFECTS INVOLVING STRONGLY INTERACTING PARTICLES

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A number of electromagnetic effects involving strongly interacting particles are discussed. The calculations are carried out within the framework of quantum electrodynamics by introducing form factors. It is shown that by studying the considered effects it is possible to obtain information on the structure and the characteristics of strongly interacting particles.

IT is well known that the absence of a theory describing strong interactions make it impossible to interpret experiments involving π-mesons and nucleons in a unique way. In particular, it seems not to be possible on the basis of such experiments to substantiate the current ideas of the electromagnetic structure of strongly interacting particles. Therefore it is of interest to investigate such electromagnetic effects involving strongly interacting particles which can be interpreted within the framework of quantum electrodynamics.

The experimental investigation of effects of this nature could give direct information on the structure and the electromagnetic characteristics of nucleons and π-mesons. However, at present it is unfortunately rather difficult to perform such experiments.

The present paper will treat the creation and annihilation of π-mesons and nucleon-antinucleon pairs in "pure" electromagnetic processes, i.e., in the case where atomic nuclei or similar strongly interacting particles are absent in the initial state. In addition, electromagnetic π-π scattering will be treated.

1. CREATION OF π-MESON AND NUCLEON-ANTINUCLEON PAIRS IN THE ANNIHILATION OF HIGH ENERGY POSITRONS

We now consider the creation of π-meson and nucleon-antinucleon pairs in positron annihilation. We shall assume that quantum electrodynamics is valid up to an energy in the center of mass system $E_{cr} \leq M$ where M is the nucleon mass[1] (the units are such that $\hbar = c = 1$). This assumption strictly speaking does not allow to discuss antinucleon creation, while the treatment of π-meson pair creation particularly at not too high energies seems to be fully justified. Therefore the possible deviations from the results obtained within the framework of quantum electrodynamics can be ascribed to the "anomalous" electromagnetic properties of the strongly interacting particles. Thus, the interaction of π-mesons with virtual nucleons will lead to a finite electromagnetic size of the particle,[2] i.e., to the form factor of the mesons. For nucleons the interaction with the real vacuum will lead in addition to the "smearing out" of the particle also to the anomalous magnetic moment. Another reason for such deviations would be a strong interaction between the particles of the created pair; this would also show up as a form factor.[3] However, as will be shown below there exists an energy region where the nature of the form factor can be uniquely established.

The smallest energy of a positron (electron) necessary for the creation of a meson pair equals μ in in the c.m.s. where μ is the mass of the π-meson. In the laboratory system this corresponds to a positron energy $E_p = 2\mu^2/m = 76$ Bev. This large difference of the energy in the two systems evidently shows that the only feasible way of performing experiments of this nature is by means of colliding beams of fast particles. The differential cross section for creation of a $\pi^\pm$ pair in the c.m.s. is given by

$$d\sigma = (\pi/16)\, r_0^2 (\mu/E)^2 (1 - \mu^2/E^2)^{3/2} \sin^2\theta\, |F|^2\, d(\cos\theta), \tag{1}$$

where E is the energy of the particle in the c.m.s. and θ is the angle between the momentum of the positron and the π^+-meson. The form factor F which is a relativistically invariant function of the energy and the momenta does not depend on the angles in the c.m.s.[3] Therefore, after integration over the angles we obtain

$$\sigma = (\pi/12)\, r_0^2 (\mu/E)^2 (1 - \mu^2/E^2)^{3/2} \,|\, F(2E^2/M_0^2) \,|^2. \tag{2}$$

The quantity M_0 in the argument of the form factor corresponds to a certain reciprocal length, characteristic for a π-meson (evidently $\mu \leq M_0 \leq M$). For $x = 0$ we have $F(x) \approx 1$. Therefore it is essential to calculate the form factor for $E \gtrsim M_0$.

At the threshold of the reaction, for E close to μ, we have

$$\sigma = (\pi/12)\, r_0^2 (1 - \mu^2/E^2)^{3/4} \,|\, F \,|^2. \tag{3}$$

We note that the annihilation of an electron-positron pair in singlet states into a $\pi^{\pm}$ pair is forbidden for reasons of conservation of parity and angular momentum. (See also Ref. 4.) On the other hand, the creation of the meson pair in an S-state is excluded since the transition matrix element is proportional to the meson momentum.[5] Therefore the smallest possible angular momentum of the meson pair is $\ell = 1$ (P-wave). This circumstance renders the interaction of the mesons in the pair unimportant at small energies in view of the short range of the interaction. It therefore follows that for small energies the form factor is uniquely given by the electromagnetic dimensions of the π-mesons.

In the relativistic case $E \gg \mu$ we have

$$\sigma = (\pi/12)\, r_0^2 (\mu/E)^2 \,|\, F \,|^2. \tag{4}$$

Neglecting the form factor the cross section (2) has its maximum at $E = 1.58\mu$. There it has a value $\sigma = 0.5 \times 10^{-31}$ cm^2. The form factor which decreases with increasing energy will shift this maximum to a smaller energy.

Equations (1) — (4) are also valid for the creation of K-meson pairs if one substitutes for μ and F respectively the mass and form factor of the K-meson.

The minimum energy of the positron (electron) in the c.m.s. for creation of a nucleon-antinucleon pair equals the nucleon mass M. In the laboratory system this corresponds to a minimum positron energy $M_p = 3.4 \times 10^3$ Bev. Here one has to take into account also the "anomalous" magnetic moment of the nucleon which can have a magnitude different from the static case. In an analogous way the cross section for creation of nucleon-antinucleon pairs can be obtained. In the c.m.s. it is given by

$$\sigma = \frac{\pi}{3}\frac{e^2}{M}\left(\frac{M}{E}\right)^2\left(1 - \frac{M^2}{E^2}\right)^{1/2}\left\{\left(1 + \frac{M^2}{2E^2}\right)\frac{e^2}{M} - 6e\mu_1 + 2\mu_1^2\frac{E^2}{M}\left(1 + \frac{2M^2}{E^2}\right)\right\} |\, F \,|^2, \tag{5}$$

where E is the positron (electron) energy, and the full magnetic moment equals $(e/2M) + \mu_1$.

In the nonrelativistic approximation $E \gtrsim M$ Eq. (5) becomes

$$\sigma = (2\pi e^2/M)(1 - M^2/E^2)^{1/2}\{e^2/4M - \mu_1 e + \mu_1^2 M\} \,|\, F \,|^2. \tag{6}$$

Putting $e = 0$ in the curly brackets of (5) and (6) we obtain the cross section for creation of a neutron-antineutron pair. As seen from these equations the anomalous magnetic moment contributes the main part to the cross section for $\mu_1 > 1$. Then the form factor is determined mainly by the basic strong interaction between the particle and the antiparticle of the pair and evidently depends much less on the electromagnetic size of the particles in contrast to the above treated case of π- and K-mesons.

2. ON THE ELECTROMAGNETIC INTERACTIONS OF π-MESONS

At present there exists considerable interest on the problem of a specific meson-meson interaction. Definite data on this interaction obviously can be obtained only in an experiment involving two colliding meson beams. It is therefore of interest to split off the purely electromagnetic part of the meson-meson interaction. To this end we shall investigate the scattering of mesons by mesons and the annihilation of a π^+ and a π^- with emission of two γ-quanta or a pair of light fermions.

The differential cross section for scattering of π-mesons of the same sign of the charge in the c.m.s. is given by

$$d\sigma = \frac{1}{16}\frac{e^4}{(pv)^2}\left[\frac{1 + v^2\cos^2(\theta/2)}{\sin^2(\theta/2)} + \frac{1 + v^2\sin^2(\theta/2)}{\cos^2(\theta/2)}\right]^2 \left| F\left(\frac{2p^2}{M_0^2}\sin^2\frac{\theta}{2}\right)\right|^2 d\Omega \tag{7}$$

where p and v are the momentum and the velocity of the mesons respectively and θ is the scattering angle. The form factor F describes the electromagnetic size of the π-mesons. Since the electromagnetic interactions are important only for small angle scattering, i.e., for large impact parameters, we shall not take into account the influence of the short range specific meson-meson interaction on the form factor.

In the limiting cases of small and large energies we have

$$d\sigma_{\text{nonrel}} = e^4 (pv)^{-2} \sin^{-4}\theta \mid F \mid^2 d\Omega, \quad d\sigma_{\text{rel}} = \frac{e^4}{p^2}\left[1 + \sin^4 \frac{\theta}{2} + \cos^4 \frac{\theta}{2}\right]^2 \mid F \mid^2 \frac{d\Omega}{\sin^4 \theta}.$$

In the nonrelativistic case (7) is identical with the first Born approximation to Mott's formula[6] derived for the scattering of slow α-particles in helium.

For mesons of opposite charge the scattering cross section is given in the c.m.s. by

$$d\sigma = \frac{1}{16} \frac{e^4}{(pv)^2}\left[\frac{1 + v^2 \cos^2 (\theta/2)}{\sin^2 (\theta/2)} - v^4 \cos\theta\right]^2 \mid F \mid^2 d\Omega. \qquad (8)$$

The first term in (8) corresponds to scattering of spin-zero particles while the second term is due to the possibility of meson annihilation.

In analogy to above we have

$$d\sigma_{\text{nonrel}} = \frac{1}{16} \frac{e^4}{(pv)^2} \frac{1}{\sin^4(\theta/2)} \mid F \mid^2 d\Omega, \quad d\sigma_{\text{rel}} = \frac{1}{16} \frac{e^4}{p^2}\left[1 + \sin^4 \frac{\theta}{2} + \cos^4 \frac{\theta}{2}\right]^2 \mid F \mid^2 \frac{d\Omega}{\sin^4(\theta/2)}.$$

Comparing (7) and (8) one sees that the differential cross sections for small angles are the same for both processes.

The differential cross sections for annihilation of two $\pi^{\pm}$-mesons into two γ-quanta is given in the c.m.s. by

$$d\sigma = r_0^2 \frac{1}{v_{12}} \left(\frac{\mu}{E}\right)^2 \left\{ {}^1\!/_2 - \frac{\mu^2}{E^2 - p^2 \cos^2\theta} + \frac{\mu^4}{(E^2 - p^2 \cos^2 \theta)^2} \right\} \mid F \mid^2 d\Omega,$$

where E is the energy of the π-mesons, v_{12} the relative velocity of the annihilating particles, and μ the π-meson mass. The form factor F is here essential because in annihilation small distances are important where the influence of the short range forces may be great. The total cross section in the c.m.s. neglecting the form factor is given by

$$\sigma = \frac{2\pi}{v_{12}} r_0^2 \left(\frac{\mu}{E}\right)^2 \left\{1 + \frac{\mu^2}{E^2} - \frac{\mu^2}{2E\sqrt{E^2 - \mu^2}}\left(2 - \frac{\mu^2}{E^2}\right) \ln \frac{E + \sqrt{E^2 - \mu^2}}{E - \sqrt{E^2 - \mu^2}} \right\}. \qquad (9)$$

In the nonrelativistic approximation $E \approx \mu$ and

$$\sigma = 2\pi r_0^2 / v_{12}. \qquad (10)$$

At large energies we have

$$\sigma = \pi r_0^2 (\mu / E)^2.$$

The total cross section for annihilation of mesons with creation of a pair of charged fermions of mass m and anomalous magnetic moment $\mu_1 = \mu_0 - e/2m$ is

$$\sigma = \frac{2\pi}{3}\left(\frac{e^2}{E}\right)^2 \left(1 - \frac{\mu^2}{E^2}\right)^{1/2} \left(1 - \frac{m^2}{E^2}\right)^{1/2} \left\{\frac{1}{2}\left(1 + \frac{m^2}{2E^2}\right) - \frac{3\mu_1}{e}\mu + \left(\frac{\mu_1}{e}\right)^2 E^2 \left(1 + \frac{m^2}{E^2}\right)\right\} \mid F \mid^2. \qquad (11)$$

Here the form factor F as shown above at not too large energies is given essentially by the electromagnetic size of the π-mesons and does not depend on the angles. Putting $\mu_1 = 0$ we have for the annihilation of $\pi^{\pm}$-mesons into an electron-positron pair

$$\sigma = (\pi e^4 / 3E^2)(1 - \mu^2 / E^2)^{1/2}. \qquad (12)$$

In the nonrelativistic approximation this becomes

$$\sigma = \pi r_0^2 v_{12} / 6. \qquad (13)$$

For relativistic energies of the annihilating particles we have

$$\sigma = (\pi r_0^2/3)(\mu / E)^2.$$

We note that Brown and Peshkin[7] have given formulae for the electromagnetic annihilation of antinucleons.

Finally we shall discuss the properties of an atomic system comprising π^+ and π^--mesons (π-mesonium). Such a system might appear in nuclear reactions producing a pair of π-mesons. Near the threshold the ratio of the number of created π-mesonium atoms to the number of free π-meson pairs is given by

$$\sigma_{\text{bound}} / \sigma_{\text{free}} \approx \alpha^3 (\mu / \Delta)^{3/2},$$

where $\alpha = 1/137$, and Δ ($\Delta \ll \mu$) is the excess energy above the reaction threshold.

From (10) one can estimate the lifetime of π-mesonium in the 1S state with respect to decay to two γ-quanta:

$$\tau_{2\gamma} = \{(\sigma v_{12})_{v_{12} \to 0} | \Psi(0) |^2\}^{-1} \approx \frac{4}{\alpha^5 \mu} \approx 0.9 \times 10^{-12} \text{ sec} \tag{14}$$

We note that in the annihilation of π-mesonium in the 1S state the polarization of the produced photons is parallel, in contradistinction to the annihilation of parapositronium where the polarization of the photons is perpendicular.

The mean life of π-mesonium against decay into an electron-positron pair can not be obtained from (14) since in the approximation used no S-waves participate in the process (see Sec. 1). In the next approximation the mean life to π-mesonium in the 1S state against decay into an electron-positron pair is of the order

$$\tau(e^+ + e^-) \sim (137)^2 \tau_{2\gamma}.$$

We also note that

$$\tau(e^+ + e^-) \sim \tau(2(e^+ + e^-)),$$

where the right hand side is the mean life of π-mesonium in the 1S state against decay into two electron-positron pairs.

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ENERGY DISTRIBUTIONS IN TWO-PARTICLE DECAYS

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Isotropic decay into two particles is treated in the laboratory reference system. It is shown that the distribution of the secondaries with respect to the quantity v defined by Eq. (42) has a number of simple properties ("symmetry properties") which are independent of the energy distribution of the primaries. Formulas are given which enable one, from the v-spectrum of one of the secondaries, to determine the v-spectra of the other secondary and of the primary particle. Symmetry properties are also found for the spectra of the products of a cascade decay. Similar formulas for the energy spectra can be obtained by a simple computation. When the angular distribution is taken into account, these formulas are approximately applicable to the high energy part of the spectrum. In addition to quantitative applications, it is pointed out that the relations derived here may be used for identification of the mass of the primaries and for establishing the isotropy or the cascade nature of the decay.

IN the two-particle breakup of an elementary particle, the energy spectrum of the decay products is determined by the kinematic features of the decay and the energy spectrum of the primary. The simplicity of this relation results in a number of general properties of the spectra of secondary particles in the laboratory reference system. For decays into two photons, these properties were explained by Carlson et al.[1] Subsequently, Sternheimer[2] solved the problem of determining the high energy part of the spectrum of the parimary particle (including its angular distribution) from the spectrum of its (arbitrary) decay products. In the present paper, after simplifying the computational procedure, we explain the internal properties of the spectrum of the secondaries. We then solve the problem of finding the energy spectrum and the mass of the primary particle from the energy spectrum of the secondary particle, as well as the problem of finding the spectrum and mass of one of the secondaries from the spectrum of the other.

1. PRIMARY FORMULA

Suppose that in a certain volume there occurs the reaction $A \rightarrow a_1 + a_2$ of decay of the particle A having mass M, momentum P and total energy E, into two particles characterized by m_1, m_2, p_1, p_2, e_1, e_2. We shall use an asterisk to denote these same quantities in the center of mass system (c.m.s.). Only the energy e_1 of the particle a_1 is recorded, without regard to its direction of motion. The following assumptions are made: the decay of A occurs isotropically; the energy distribution of A is given in the energy interval from $E_{\text{min}} = M$ to E_{max}; all the particles a_1 are recorded; the statistics are sufficiently good so that the shape of the energy distribution is precisely known.

The distribution of A in energy and angle is given by $\overline{N}(E, \vartheta)\, dE\, d\cos\vartheta$. We know that, for the case of a two-particle decay with isotropic distribution of the decay products in the c.m.s., the energy distribution of the secondaries for a fixed energy of the primary is equal to the constant $(M/2Pp^*)de_1$, if e_1 satisfies the inequality $e_{1\,\text{min}} \le e_1 \le e_{1\,\text{max}}$, and is equal to zero outside of the interval $(e_{1\,\text{min}}, e_{1\,\text{max}})$. The energy e_1 reaches its largest (smallest) value when the particle a emerges forward (backward) relative to the motion of particle A. Lorentz transformations give

$$\begin{matrix} e_{1\,\text{min}} \\ e_{1\,\text{max}} \end{matrix} = \frac{Ee_1^* \mp Pp^*}{M}. \tag{1}$$

The number of particles with energy e_1 in the interval of width de_1, produced from particles A with energy in the interval dE and direction in $d\vartheta$, is

$$(M/2Pp^*)\overline{N}(E,\ \vartheta)\,dE\,d\cos\vartheta\,de_1,$$

while the number of particles a_1, formed from particles A with arbitrary ϑ, is

$$(M/2Pp^*)\widetilde{N}(E)\,dE\,de_1,$$

where $\widetilde{N}(E) = \int\limits_{\vartheta} \overline{N}(E, \vartheta)\,d\cos\vartheta$ is the average of the energy spectrum of A over all directions. In addition, the number density of particles a_1 in the energy interval de_1, formed from the decay of particles A of arbitrary energy, will be

$$\widetilde{n}_1(e_1) = \int\limits_{E_{\min}}^{E_{\max}} (M/2Pp^*)\widetilde{N}(E)\,dE, \qquad (2)$$

where $E_{\min}$ and $E_{\max}$ are the smallest and largest energies of particles A which produce particles with energy e_1.

We proceed to simplify Eq. (2). To do this we shall characterize the motion of the particles by positive quantities u or v according to the formulas

$$E = M\cosh u, \quad e_{1,2} = m_{1,2}\cosh v_{1,2}. \qquad (3)$$

It follows from (1) that under Lorentz transformations the arguments for particles a_1 which move forward (backward) are expressed simply as:

$$v_{1\max} = v_1^* + u \text{ (forward)}, \quad v_{1\min} = |\,v_1^* - u\,| \text{ (backward)}. \qquad (4)$$

Consequently, as u increases from 0 to ∞, $v_{1\max}$ increases from v_1^* to ∞; $v_{1\min}$ decreases from v_1^* to 0 in the interval where u increases from 0 to v_1^*, and then increases from 0 to ∞ when u goes from v_1^* to ∞. From (4) it follows that particles a_1 with argument $v_1 \geq v_1^*$ arise only from particles A with arguments in the interval $(v_1 - v_1^*,\ v_1 + v_1^*)$, while particles a_1 with argment $v_1 \leq v_1^*$ come from particles A with arguments in the interval $(v_1^* - v_1,\ v_1 + v_1^*)$. Equation (2) can therefore be written as

$$n_1(v_1) = k \int\limits_{|v_1 - v_1^*|}^{v_1 + v_1^*} N(u)\,du, \qquad (5)$$

where

$$k = M/2p^*, \quad \text{a} \quad n_1(v) \equiv \widetilde{n}_1(m_1\cosh v), \ N(u) \equiv \widetilde{N}(M\cosh u)$$

are the "densities" for the energy distributions of particles a_1 and A converted to the arguments v and u. It should be remembered that to get, say, the number of particles with arguments between α and β, $n_1(v)$ must be weighted by the factor $m_1 \sinh v$:

$$\int\limits_{\alpha}^{\beta} m_1 \sinh v\, n_1(v)\,dv.$$

The transformations (3) are not applicable when the rest mass is zero. However, if we map the half-line $e_1 > 0$ onto the line v_1 by means of

$$e_1 = e_1^* \exp v_1, \qquad (6)$$

then the Lorentz transformations take the form $v_{1\max} = u$ (forward), $v_{1\min} = -u$ (backward), and the relation into which (2) goes for $m_1 = 0$,

$$\widetilde{n}_1(e_1) = \int\limits_{E_{\min}}^{\infty} (M/2Pp^*)\widetilde{N}(E)\,dE$$

becomes

$$n_1(v_1) = k \int\limits_{|v_1|}^{\infty} N(u)\,du. \tag{7}$$

Here we have

$$n_1(v_1) \equiv \tilde{n}_1(e_1^* \exp v_1).$$

The simple form of (5), (7) makes them convenient for further calculations.

2. RELATION TO THE MASS OF THE PRIMARY PARTICLE

Let $N(u) \equiv 0$ for $u \geq U$, where $U \leq v_1^*$. Then the upper limit in (5) is U. Since the integrand $N(u)$ is non-negative, it follows from (5) that $n_1(v_1)$ is a monotonically decreasing function of the lower limit $|v_1 - v_1^*|$. Its maximum value is attained for $v_1 = v_1^*$, and $n_1(v_1) \leq n_1(v_1^*)$ for all $v_1 \neq v_1^*$. Thus we find property A: the spectrum of the secondary particle, in the case where the arguments of particle A do not exceed $U \leq v_1^*$ (i.e., its energy is no greater than $E^{(1)} = Me_1^*/m_1$), has a maximum (peak or plateau) at the point with abscissa $v_1 = v_1^*$. The spectrum of gamma quanta shows a similar property at $v_1 = 0$ for any energy range of the primary particles.*

Since n_1 depends only on $|v_1 - v_1^*|$, we have property B—the spectrum $n_1(v_1)$ is symmetric around the point v_1^*:

$$v_1^* = (v' + v'')/2, \tag{8}$$

where v' and v'' are the abscissas of points in the spectrum with equal ordinates.

Similarly, the spectrum of gamma quanta has the property

$$v' + v'' = 0. \tag{8$'$}$$

Properties A and B enable us to determine the mass of one of the particles if we know the masses of the other two in the reaction, since v_1^* is given in terms of the masses of the particles by

$$(M^2 + m_1^2 - m_2^2)/2M = m_1 \cosh v_1^*. \tag{9}$$

The expression for M has the form

$$M/m_1 = \cosh v_1^* + [\sinh^2 v_1^* + (m_2/m_1)^2]^{1/2}. \tag{10}$$

For $m_1 = m_2 = m$, Eq. (10) reduces to

$$M = 2m \cosh v_1^*. \tag{11}$$

The set (10) and (8) determine M either from the position of the peak or from the abscissas of the points on a horizontal line. Returning to the usual energy spectra gives

$$M = [(e_1'e_1'' + p_1'p_1'' + m_1^2)/2]^{1/2} + [e_1'e_1'' + p_1'p_1'' + 2m_2^2 - m_1^2)/2]^{1/2}; \tag{10$'$}$$

or, for $m_1 = m_2 = m$:

$$M = [2(e_1'e_1'' + p_1'p_1'' + m^2)]^{1/2}, \tag{11$'$}$$

which reduces, for $m = 0$, to the formula $M = 2(e_1'e_2')^{1/2}$ derived in Ref. 1. From (8$'$) or (10$'$) we can also get the formula for the case of $m_1 = 0$, $m_2 \neq 0$:

$$M = (e_1'e_1'')^{1/2} + (e_1'e_1'' + m_2^2)^{1/2}. \tag{12}$$

Formulas (10), (10$'$), (11), and (11$'$) are valid in the absence of primary particles with energies above $E^{(1)} = Me_1^*/m_1$. Later we shall weaken these restrictions. Formula (12) is valid without any restriction of this sort.

Property B might be called the logarithmic symmetry of the spectrum of secondary particles,[4] since the argument is given in terms of the logarithm

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*We note that the condition $u = v_1^*$ has a simple physical meaning: for values of u higher than v_1^*, the secondary particle can only move forward in the laboratory system, i.e., there is a limiting angle of emergence. For $u < v_1^*$ there is no limiting angle.

$$v_1 = \ln[(e_1 + p_1)/m_1] \qquad (m_1 \neq 0), \tag{13}$$

$$v_1 = \ln(e_1/e_1^*) \qquad (m_1 = 0). \tag{13'}$$

For non-relativistic energies, v_1 coincides with the velocity of the particle.

Property B can also be used to determine m_2 when m_1 and M are known:

$$m_2 = (M^2 + m_1^2 - 2Mm_1 \cosh v_1^*)^{1/2}. \tag{14}$$

3. DETERMINATION OF THE SPECTRUM OF THE PRIMARY PARTICLE

We differentiate (5) with respect to v_1 for $v_1 \geq v_1^*$:

$$k^{-1} n_1'(v_1) = N(v_1 + v_1^*) - N(v_1 - v_1^*).$$

To solve this difference equation for N, we set $v_1 - v_1^* = u$ and give u successive values $u + 2v_1^*$, $u + 4v_1^*$, ... Summing these equations gives†

$$N(u) = -k^{-1} \sum_{\nu=0}^{\nu'} \tilde{n}_1' [u + (2\nu + 1) v_1^*]. \tag{15}$$

The summation ends when $n_1' = 0$. Thus the number density of primary particles with argument u is proportional to the sum of the derivatives of the spectrum of secondaries at points with abscissas forming an arithmetical progression with difference $2v_1^*$ and initial term $u + v_1^*$. In the usual notation, (15) has the form

$$\widetilde{N}(E) = -(2p^*/M) \sum_{\lambda} (\varepsilon_\lambda P + \rho_\lambda E)\, \tilde{n}_1' (\varepsilon_\lambda E + \rho_\lambda P), \tag{15'}$$

where

$$\varepsilon_\lambda = \frac{m_1}{M} \cosh \lambda v_1^*, \quad \rho_\lambda = \frac{m_1}{M} \sinh \lambda v_1^*, \lambda = 1, 3, 5, \ldots, \tag{16}$$

while v_1^* is given by Eq. (9).

The high energy part of the spectrum (where $E \simeq P$) can be determined from the equation

$$\widetilde{N}(E) \simeq -(2m_1 p^*/M^2)\, E \sum_{\lambda} \exp(\lambda v_1^*)\, \tilde{n}_1' (E \exp \lambda v_1^*). \tag{17}$$

As an example, we consider the reaction $\theta^0 \to 2\pi$. Here $v_\pi^* = 1.18$, so that

$$\widetilde{N}(E) = -0.829 \{(0.5P + 0.415E)\, \tilde{n}'(0.5E + 0.415P) + 4.84 (P + E)\, \tilde{n}' [4.84 (P + E)] + \cdots\},$$

and for large E,

$$\widetilde{N}(E) \simeq -0.829 E\, [0.915\, \tilde{n}'(0.915E) + 9.69\, \tilde{n}'(9.69E) + \ldots]. \tag{18}$$

The last formula coincides with (42a) of Ref. 2, although it was derived under different assumptions. Later we shall explain the reason for this coincidence.

For the case of $m_1 = 0$, the formula for N was derived in Ref. 1. In our notation it has the form

$$N(u) = -k^{-1} n_1'(u) \tag{19}$$

and is valid even for $m_2 \neq 0$.

4. THE SPECTRUM OF THE OTHER SECONDARY PARTICLE

By substituting in the expression for the spectrum of a_2 $(m_2 \neq 0)$:

$$n_2(v_2) = k \int_{|v_2 - v_2^*|}^{v_2 + v_2^*} N(u)\, du$$

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†The prime denotes differentiation with respect to the argument.

the spectrum of A as obtained from (15) or (19), we can construct the spectrum of a_2 from that of a_1:

$$n_2(v_2) = \sum_\lambda [n_1(|v_2 - v_2^*| + \lambda v_1^*) - n_1(v_2 + v_2^* + \lambda v_1^*)] \quad (m_1,\ m_2 \neq 0), \tag{20}$$

$$n_2(v_2) = n_1(|v_2 - v_2^*|) - n_1(v_2 + v_2^*) \quad (m_1 = 0,\ m_2 \neq 0). \tag{21}$$

(In all sums, $\lambda = 1, 3, 5, \ldots$)

If $m_2 = 0$, the spectrum of a_2 is expressible in terms of the spectrum of a_1 by means of (7) and (15):

$$n_2(v_2) = \sum_\lambda n_1(|v_2| + \lambda v_1^*) \quad (m_1 \neq 0,\ m_2 = 0). \tag{22}$$

Reverting to the usual parameters, energy and momentum, we rewrite (20) – (22) as

$$\begin{aligned} \widetilde{n}_2(e_2) &= \sum_\lambda [\widetilde{n}_1(\alpha_\lambda^+ e_2 - \beta_\lambda^+ p_2) - \widetilde{n}_1(\alpha_\lambda^+ e_2 + \beta_\lambda^+ p_2)] \quad (e_2 \leqslant e_2^*), \\ \widetilde{n}_2(e_2) &= \sum_\lambda [\widetilde{n}_1(\alpha_\lambda^- e_2 + \beta_\lambda^- p_2) - \widetilde{n}_1(\alpha_\lambda^+ e_2 + \beta_\lambda^+ p_2)] \quad (e_2 \geqslant e_2^*), \end{aligned} \tag{20'}$$

where $m_1,\ m_2 \neq 0$, and

$$\begin{matrix} \alpha_\lambda^\pm \\ \beta_\lambda^\pm \end{matrix} = \frac{m_1}{m_2} \begin{matrix} \cosh \\ \sinh \end{matrix} (\pm v_2^* + \lambda v_1^*). \tag{23}$$

For $m_1 = 0,\ m_2 \neq 0$:

$$\begin{aligned} \widetilde{n}_2(e_2) &= \widetilde{n}_1[e_1^* M(e_2 - p_2)/m_2^2] - \widetilde{n}_1[e_1^* M(e_2 + p_2)/m_2^2] \quad (e_2 \leqslant e_2^*), \\ \widetilde{n}_2(e_2) &= \widetilde{n}_1[e_1^*(e_2 + p_2)/M] - \widetilde{n}_1[e_1^* M(e_2 + p_2)/m_2^2] \quad (e_2 \geqslant e_2^*). \end{aligned} \tag{21'}$$

For $m_1 \neq 0,\ m_2 = 0$:

$$\begin{aligned} \widetilde{n}_2(e_2) &= \sum_\lambda \widetilde{n}_1[(m_1 e^{-\lambda v_1^*} e_2/2e_2^*) + (m_1 e^{\lambda v_1^*} e_2^*/2e_2)] \quad (e_2 \leqslant e_2^*), \\ \widetilde{n}_2(e_2) &= \sum_\lambda \widetilde{n}_1[(m_1 e^{\lambda v_1^*} e_2/2e_2^*) + (m_1 e^{-\lambda v_1^*} e_2^*/2e_2)] \quad (e_2 \geqslant e_2^*). \end{aligned} \tag{22'}$$

We give some particular cases of these formulas. For the reaction $\Xi^- \to \Lambda^0 + \pi^-$, $e_\Lambda \geq 1.1192$ Bev,

$$n_\Lambda(e_\Lambda) = n_\pi(0.17\, e_\Lambda + 0.11\, p_\Lambda) - n_\pi(0.20\, e_\Lambda + 0.15\, p_\Lambda) + n_\pi(0.81\, e_\Lambda + 0.80\, p_\Lambda) - n_\pi(1.05\, e_\Lambda + 1.04\, p_\Lambda) + \ldots$$

For the reaction $\Sigma^0 \to \Lambda^0 + \gamma$:

$$n_\Lambda(e_\Lambda) = n_\gamma(0.065\,(e_\Lambda + p_\Lambda)) - n_\gamma(0.075\,(e_\Lambda + p_\Lambda)) \quad (e_\Lambda \geqslant 1.1126 \text{ Bev}),$$

$$n_\gamma(e_\gamma) = n_\Lambda(6.7\, e_\gamma + 0.046\, e_\gamma^{-1}) + n_\Lambda(5.8\, e_\gamma + 0.053\, e_\gamma^{-1}) + \ldots \quad (e_\gamma \leqslant 0.0773 \text{ Bev}),$$

$$n_\gamma(e_\gamma) = n_\Lambda(7.7\, e_\gamma + 0.040\, e_\gamma^{-1}) + n_\Lambda(8.9\, e_\gamma + 0.035\, e_\gamma^{-1}) + \ldots \quad (e_\gamma \geqslant 0.0773 \text{ Bev}).$$

5. INTERNAL PROPERTIES OF THE SPECTRA OF SECONDARY PARTICLES

Equation (15) shows that the spectrum of A can be determined from that part of the spectrum of a_1 for which $v_1 \geq v_1^*$. On the other hand, the spectrum of A determines the whole spectrum of a_1, including the region $v_1 \leq v_1^*$. There must therefore be a relation between the regions $v_1 \leq v^*$ and $v_1 \geq v^*$.† In fact, substituting (15) in (5) and setting $v_1 \leq v_1^*$, we get (property C):

$$n_1(v_1) = n_1(2v_1^* - v_1) - n_1(2v_1^* + v_1) + n_1(4v_1^* - v_1) - n_1(4v_1^* + v_1) + \ldots \tag{24}$$

In Sec. 2 we presented a method for determining v_1^* and indicated the limiting energy up to which this procedure was applicable. Formula (24) enables us, at least in principle, to reduce and finally to eliminate this restriction on the energy of the primary particle.

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†M. I. Podgoretskii called my attention to this point.

The arguments of the successive terms on the right side of (24) vary respectively within the intervals $(v_1^*, 2v_1^*)$, $(2v_1^*, 3v_1^*)$, etc. If $U \le v_1^*$, then $v_{1\,\max} \le 2v_1^*$, and all terms in (24) except the first vanish, so that we are left with the equation

$$n_1(v_1) = n_1(2v_1^* - v_1) \tag{25}$$

which is a statement of property B (cf. above). Next, let $v_1^* \le U < 2v_1^*$. Then $v_{1\,\max} = U + v_1^* < 3v_1^*$, and we must take two terms on the right of (24). However, for sufficiently large v_1 the second term is identically zero. This begins when the inequality $2v_1^* + v_1 \ge U + v_1^*$ is satisfied, so that $v_1 \ge U - v_1^*$, and the argument of the first term on the right is $3v_1^* - U$. We finally get the following result: if $v_1^* \le U < 2v_1^*$, then (25) is valid in the neighborhood of v_1^*, in the interval $(U - v_1^*, 3v_1^* - U)$; outside of this interval, in the neighborhood of 0 and $2v_1^*$, i.e., in the intervals $(0, U - v_1^*)$, $(3v_1^* - U, U + v_1^*)$, we have the equation

$$n_1(v_1) = n_1(2v_1^* - v_1) - n_1(2v_1^* + v_1). \tag{26}$$

Having found three points v', v'', v''' such that (26) is satisfied [i.e., $n_1(v') = n_1(v'') - n_1(v''')$] and for which $2v' = v''' - v''$, we determine $v_1^* = (v' + v'')/2$ (property C; see Fig. 1).

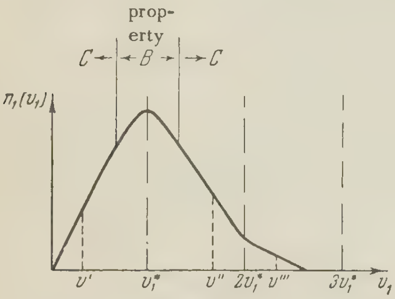

FIG. 1

Similarly, we can show that if $2v_1^* \le U < 3v_1^*$, then in the neighborhoods of 0 and $2v_1^*$, in the intervals $(0, 3v_1^* - U)$, $(U - v_1^*, 5v_1^* - U)$, Eq. (26) is satisfied, while outside, in the neighborhood of v_1^* and $3v_1^*$, i.e., in the intervals $(3v_1^* - U, U - v_1^*)$, $(5v_1^* - U, U + v_1^*)$, the condition is

$$n_1(v_1) = n_1(2v_1^* - v_1) - n_1(2v_1^* + v_1) + n_1(4v_1^* - v_1) \tag{27}$$

etc.

Thus, from $U \sim 0$ up to $U^{(2)} = 2v_1^*$ there is a portion of the spectrum where property B manifests itself, while from $U^{(1)} = v_1^*$ up to $U^{(3)} = 3v_1^*$ there is a part of the spectrum showing property C′, etc. The limitation on the energy of the primary particle which was given in Sec. 1 is thus removed.

Limiting Energies of Primary Particle (Bev)

Reaction	Spectrum	$E^{(1)}$	$E^{(2)}$	$E^{(3)}$
$\pi \to \mu + \nu$	μ	0.146	0.16	0.19
$\theta^0 \to 2\pi$	π	0.85	2.5	7.7
$\Lambda^0 \to p + \pi^-$	p	1.115	1.13	1.16
$\Lambda^0 \to p + \pi^-$	π	1.33	2.1	3.6
$\Sigma^+ \to p + \pi^0$	π	1.97	5.3	16
$\Xi^- \to \Lambda^0 + \pi^-$	π	1.88	4.0	9.6
$\Sigma^0 \to \Lambda^0 + \gamma$	γ	∞	—	—

In the table, we give the limiting values of the energy, $E^{(1)}$, $E^{(2)}$, $E^{(3)}$ corresponding to $U^{(1)}$, $U^{(2)}$, $U^{(3)}$ for several decay reactions. For light secondaries $(m_1 \ll M)$ these limits are exceedingly high and go far out in the relativistic region.

The spectrum of a_1 has an integral property (property D). Integrating (24) between the limits $v_1 = 0$ and $v_1 = v_1^*$, and making a change of variables of the form $2\nu v_1^* \pm v_1 = t$ in each of the integrals, we get

$$\int_0^{v_1^*} n_1(v_1)\,dv_1 - \int_{v_1^*}^{2v_1^*} n_1(v_1)\,dv_1 + \int_{2v_1^*}^{3v_1^*} n_1(v_1)\,dv_1 - \ldots = 0, \tag{28}$$

i.e., if we break up the spectrum of the secondary particle into pieces of length v_1^* (where the last piece may have a shorter length), then the sum of the areas of the even-numbered pieces is equal to the sum of the areas of the odd-numbered pieces. By finding a length v, such that property D is satisfied, we determine $v_1^* = v$ and M to an accuracy exceeding that attainable by using properties A to C.

Formula (28) can also be expressed in the usual notation as

$$\Big[\int_0^{e_1^*} - \int_{e_1(v_1^*)}^{e_1(2v_1^*)} + \int_{e_1(2v_1^*)}^{e_1(3v_1^*)} - \ldots\Big]\frac{\widetilde{n}_1(e_1)}{p_1}\,de_1 = 0, \tag{29}$$

where $e_1(x) = m_1 \cosh x$.

6. THE SPECTRUM OF THE PRODUCTS OF A CASCADE DECAY

Let us consider the case where one of the products of a decay $A \rightarrow a_1 + a_2$ in turn decays into two particles, $a_1 \rightarrow \alpha_1 + \alpha_2$. Examples of this are the cascades

$$\Xi^- \rightarrow \pi^- + \Lambda^0 \rightarrow \pi^- + (\pi^- + p) \text{ or } \rightarrow \pi^- + (\pi^0 + n), \quad \Xi^0\,(\Sigma^0) \rightarrow \Lambda^0 + \pi^0 \rightarrow \Lambda^0 + 2\gamma, \ \Lambda^0 \rightarrow n + \pi^0 \rightarrow n + 2\gamma.$$

In the last three cascades, for low energies, the two decay processes occur practically at one point. The spectrum of a_1 has axial symmetry, so that the spectrum of α_1 should have some additional symmetry. In order to reveal this additional property of the energy spectrum of the products of a cascade decay, let us assign to the particle α_1 the argument w with the characteristic value w^*, and assume first that the mass of α_1 is not equal to zero. The distribution of α_1 with respect to the argument w will be

$$\nu(w) = \int\limits_{|w-w^*|}^{w+w^*} k_2 dv \int\limits_{|v-v_1^*|}^{v+v_1^*} k_1 N(u)\, du, \tag{30}$$

where

$$k_2 = m_1 / 2 m_{\alpha_1} \sinh w^*, \quad k_1 = M / 2 p^*.$$

Let us assume that $v_1^* < w^*$. This condition is satisfied for all the cascade processes cited above. Also, let us first assume that $w \geq w^* + v_1^*$. Then the absolute value signs in (30) can be dropped. Differentiating (30) twice, we get:

$$(k_2 k_1)^{-1} \nu''(w) = N(w + w^* + v_1^*) - N(w + w^* - v_1^*) - N(w - w^* + v_1^*) + N(w - w^* - v_1^*).$$

The procedure for solving this difference equation is a repetition of the computations we did earlier, and leads to

$$N(u) = (k_1 k_2)^{-1} \sum_{m=0}^{m'} \sum_{n=0}^{n'} \nu''(u + k_n w^* + l_m v_1^*), \quad k_n = 2^{n+1} - 1, \ l_m = 2^{m+1} - 1; \ m, n = 0, 1, 2, \ldots \tag{31}$$

Now that we have determined the spectrum of A from the piece $w \geq w^* + v_1^*$ of the spectrum of α_1 by using (31), we can relate other portions of the spectrum of α_1 to this one. Setting $k_n w^* + \ell_m v_1^* = \lambda$ and and substituting (31) into (30), we get:

$$\nu(w) = \sum_{\nu} \int\limits_{|w-w^*|}^{w+w^*} dv_1 \int\limits_{|v_1-v_1^*|}^{v_1+v_1^*} \nu''(u + \lambda)\, du. \tag{32}$$

First let $w^* \leq w \leq w^* + v_1^*$. Then the computations give

$$\nu(w) = \sum_{\lambda} [2\nu(\lambda) + \nu(w + w^* + v_1^* + \lambda) - \nu(w - w^* + v_1^* + \lambda) - \nu(w + w^* - v_1^* + \lambda) - \nu(-w + w^* + v_1^* + \lambda)]. \tag{33}$$

Next, in the interval $w^* - v_1^* \leq w \leq w^*$ Eq. (33) is again satisfied. Finally, for $w \leq w^* - v_1^*$,

$$\nu(w) = \sum_{\lambda} \nu[(w + w^* + v_1^* + \lambda) - \nu(w - v^* + v_1^* + \lambda) - \nu(-w + w^* + v_1^* + \lambda) + \nu(-w - w^* + v_1^* + \lambda)].$$

The complexity of the formulas is an obstacle to a demonstration of the symmetry of the spectrum. However, if the spectrum of A cuts off at $U \leq v_1^*$, and as a consequence of this the spectrum of α_1 is cut off at $U + w^* + v_1^* \leq w^* + 2v_1^*$, then in (31) all that remains of the sum is the first term $\nu''(u + w^* + v_1^*)$, while in (33) we are left with $\nu(w) = 2\nu(w^* + v_1^*) - \nu(-w + 2w^* + 2v_1^*)$, which can be written in the form

$$[\nu(w) + \nu(2w^* + 2v_1^* - w)]/2 = \nu(w^* + v_1^*) \tag{34}$$

which shows that the pieces of the spectrum in the intervals $(w^*, w^* + v_1^*)$ and $(w^* + v_1^*, w^* + 2v_1^*)$ are transformed into one another under inversion in the point $(w^* + v_1^*, \nu(w^* + v_1^*))$ (see Fig. 2). If in addition $2v_1^* \leq w^*$, then $v_{1\,\max} < w^*$, and the spectrum of α_1 should also possess axial symmetry with respect to the line $w = w^*$. (The image is shown as a dotted curve in Fig. 2.) As we extend the range of the arguments, the connection between the parts of the spectrum becomes complicated.

We must also consider the case where the mass of α_1 is equal to zero. Considerations entirely analogous to those used above give

$$N(u) = (k_1 k_2)^{-1} \sum_n \nu''(u + (2n+1)v_1^*),$$

from which we get the following connection between the parts of the spectrum: for $-v_1^* \leq w \leq v_1^*$,

$$\nu(w) = [2\nu(v_1^*) - \nu(-w + 2v_1^*) - \nu(w + 2v_1^*)] + [2\nu(3v_1^*) - \nu(-w + 4v_1^*) - \nu(w + 4v_1^*)] + \ldots; \qquad (35)$$

for $w \leq -v_1^*$:

$$\nu(w) = \nu(-w).$$

If, as was assumed above, the spectrum of A ends at $U \leq v_1^*$, then (35) reduces to

$$\nu(w) = 2\nu(v_1^*) - \nu(2v_1^* - w), \qquad (36)$$

i.e., the point $(v_1^*, \nu(v_1^*))$ is a center of symmetry for the portions of the spectrum in the intervals $(0, v_1^*)$, $(v_1^*, 2v_1^*)$. It is easy to see that the point $(-v_1^*, \nu(-v_1^*))$ also has this property (see Fig. 3).

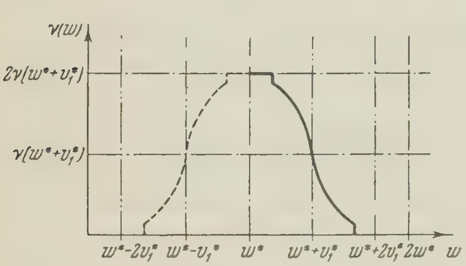

FIG. 2

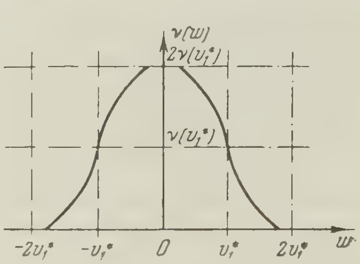

FIG. 3

If the maximum value U satisfies the condition $v_1^* \leq U \leq 2v_1^*$, the central symmetry disappears and (36) becomes more complicated:

$$\nu(w) = 2\nu(v_1^*) - \nu(2v_1^* - w) - \nu(2v_1^* + w).$$

The properties we have found may make it possible to distinguish the cascade decay $A \rightarrow a_1 + a_2 \rightarrow (\alpha_1 + \alpha_2) + a_2$ from the three-particle decay $A \rightarrow \alpha_1 + \alpha_2 + a_2$. We may expect that in the latter the spectrum of α_1 will not posses symmetry because of the anisotropy of the decay. From the location of the center of symmetry of the spectrum we can also estimate the mass of the primary particle.

Now let us consider the reaction $\pi^- + p \rightarrow \Sigma^0 + K^0 \rightarrow (\Lambda^0 + \gamma) + K^0$. The directions of Σ^0 and K^0 will not be distributed isotropically in the c.m.s. so that we should not expect any symmetry in the spectra of Σ^0 and K^0. However, the decay $\Sigma^0 \rightarrow \Lambda^0 + \gamma$ can occur isotropically, so that the spectra of Λ^0 and γ have the usual symmetry properties A to D of one-stage decay processes. The detection of such properties in the spectra of Λ^0 and γ may serve as an indication of the cascade character of the process. In addition, if the spectrum of Λ^0 is symmetric, we cannot admit any production of Λ^0 via the process $\pi^- + p \rightarrow \Lambda^0 + K^0$, which is anisotropic. Obviously, similar considerations can be applied to other reactions in which particles are created and which culminate in the decay of one of the produced particles.

7. EXTENSION OF VALIDITY OF FORMULAS

Throughout all of the preceding, the angular distributions of the secondary particles was not taken into account. It is possible to simplify Sternheimer's derivation[2] of the formulas relating the angular and energy spectra of the high energy primary and secondary particles. This will enable us to extend some of the results obtained above to the case where the secondary particle is observed in a definite direction.

Suppose that the particles a_1 all appear at the same point. Also let us assume that a definite direction of motion ϑ' is selected, and that we record the energies of particles a_1 moving in this direction. We take this direction as the polar axis. We shall assume that the distribution is uniform in azimuth. The probability for particle a_1 to have a direction in $d\vartheta'$ is

$$\frac{M}{2Pp^*} \frac{de_1}{d\cos(\vartheta - \vartheta')} d\cos\vartheta'.$$

Therefore the total number of particles moving in the direction $\vartheta' = 0$ will be

$$d\cos\vartheta' \int_E \int_\vartheta \overline{N}(E,\vartheta)\,dE\,d\cos\vartheta\ \frac{M}{2Pp^*}\ \frac{de_1}{d\cos\vartheta}\ .$$

If we require that $\cos\vartheta'$ and e_1 lie in the intervals $(1, 1 + d\cos\vartheta')$ and $(e_1, e_1 + de_1)$, then we must either treat E as a function of e_1 and ϑ or ϑ as a function of e_1 and E. The first assumption was made in Ref. 2. The number density of particles then takes the form

$$\tilde{n}_1(e_1, 0) = \int_\vartheta \overline{N}\,[E(e_1,\vartheta),\vartheta]\,\frac{dE(e_1,\vartheta)}{de_1}\ \frac{M}{2Pp^*}\ \frac{de_1(\vartheta)}{d\cos\vartheta}\,d\cos\vartheta,$$

which necessitates further messy calculations. But on the second assumption, $\vartheta = \vartheta(E, e_1)$, we immediately get

$$\tilde{n}_1(e_1 0) = (M/2p^*) \int_{E_{\min}}^{E_{\max}} \overline{N}\,[E, \vartheta(E, e_1)]\,p^{-1}\,dE.$$

This formula is essentially the same as (34) of Ref. 2, but is exact. Transforming to arguments u, v_1 in place of E, e_1, we can express the distribution with respect to u and v_1 as

$$n_1(v_1, 0) = k \int_{|v_1 - v_1^*|}^{v_1 + v_1^*} N(u, \vartheta(u, v_1))\,du, \tag{37}$$

where

$$\vartheta(u, v_1) = \arccos\,[(\cosh u \cosh v_1 - \cosh v_1^*)/\sinh u \sinh v_1]; \qquad n_1(v_1, 0) \equiv \tilde{n}_1(m_1 \cosh v_1, 0);\ N(u,\vartheta) \equiv \overline{N}(M \cosh u, \vartheta).$$

For high energies (large values of u and v_1), the angle of emergence ϑ is close to zero, and (37) can be replaced by the approximate equation

$$n_1(v_1, 0) = k \int_{|v_1 - v_1^*|}^{v_1 + v_1^*} N(u, 0)\,du. \tag{38}$$

This equation is identical with (5). Therefore all the consequences of formula (5) which are not dependent on the condition that the energy be low are also valid for (38). So formulas (15), (17), (20) — (22), (20′) — (22′) are also valid for the case where the secondary particles are observed at a definite angle. Now they give the spectra of A or a_1 in the direction of observation, and not their spectra averaged over all angles. This explains why Eq. (18) of this paper coincides with (42a) of Ref. 2.

We present formulas, valid for high energies, which give the spectrum of one of the secondaries in terms of the spectrum of the other:

$$\tilde{n}_2(e_2) = \sum_\lambda\,[\tilde{n}_1(e_2 e^{-v_2^* - \lambda v_1^*}) - \tilde{n}_1(e_2 e^{v_2^* + \lambda v_1^*})] \quad (m_1, m_2 \neq 0); \tag{39}$$

$$\tilde{n}_2(e_2) = \tilde{n}_1(2e_1^* e_2/M) - \tilde{n}_1(2e_1^* M e_2/m_2^2) \quad (m_1 = 0, m_2 \neq 0), \tag{40}$$

$$\tilde{n}_2(e_2) = \sum_\lambda \tilde{n}_1(m_1 e^{\lambda v_1^*} e_2/2e_2^*) \quad (m_1 \neq 0 \quad m_2 = 0). \tag{41}$$

These formulas are applicable when the limiting angles of emergence of both secondary particles are small.

8. CONCLUDING REMARKS

Thus, in a two-particle decay, to make clear the connection between the energy spectrum of the decay products and the energy spectrum of the primary component (averaged over all angles), it is convenient* to introduce the parameter

$$v = \cosh^{-1}\gamma = \cosh^{-1}(e/m) = \ln\frac{e+p}{m}. \quad (42)$$

The distribution of the decay products with respect to this parameter (the "v-spectrum") has simple properties which are independent of the energy distribution of the primary particles: (A) In a certain range of energies of the primary particles, the distribution of the secondaries has a maximum at a point v^* which is determined by the masses of the particles. (B) In this same energy range, the distribution of secondary particles is symmetric with respect to the vertical line through the point of the maximum. (C) When the energy range is extended, we retain the simple relation (24) between the points of the spectrum in even $(2\nu v^*, 2\nu v^* + v^*)$ and odd $(2\nu v^* + v^*, 2\nu v^* + 2v^*)$ intervals of the v-spectrum. (D) The total area under the v-spectrum in the even sections is equal to the total area of the odd sections.

In principle, the use of properties A to D enables us to identify the mass of the primary particles. These properties of v-spectra occur under the following assumptions: presence of only two decay products; isotropy of the decay in the center of mass system; absence of competing decay processes; limitation of the range of energies of the primary particles; absence of systematic errors in counting of secondary particles; absence of any preferred direction of observation; good statistics. We shall assume that those assumptions are fulfilled which depend on the arrangement of the experiment, and limit ourselves to the first two assumptions. Nothing in this paper shows that these are necessary conditions for the symmetry of the v-spectra. However, from physical considerations, it does follow with a definite probability. We can assign this same degree of reliability when (subject to the fulfillment of the other conditions) we apply the symmetry criteria to prove the isotropy or cascade character of a decay (as was recommended in Sec. 6).

The formulas for determining the energy spectra of the particles are suitable for use in experiments in emulsions or chambers where it is possible to avoid selection of a definite direction of observation. However, the existence[3-5] for certain angular distributions of "isotropic" directions, along which the flux of particles is the same as for an isotropic distribution, enables us to use these formulas in counter experiments also.

I express my gratitude to M. I. Podgoretskii for his interest in the work.

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*Cf. also Ref. 6.

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ELECTROMAGNETIC-FIELD SHOCK WAVES AND THEIR CUMULATION

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We consider electromagnetic waves with narrow transition regions between the initial and final states, and in particular, a converging cylindrical wave. It is found that as such a wave converges its amplitude increases without bound. A qualitatively new phenomenon, cumulation, is found. This is the occurrence of infinitely strong fields, at finite distances from the axis, on the front of the wave reflected from the cylindrical axis. This property is not peculiar to electromagnetic phenomena, but is related to the cylindrical geometry. Acoustical waves have the same property, but for them this solution is valid only for weak waves, whereas this restriction does not apply to electromagnetic-field waves.

1. FORMATION OF WAVES AND THICKNESS OF THE WAVE FRONT

CONSIDER semi-infinite space filled with a perfect conductor and bounded (along the $x = 0$ plane) by a vacuum with a stationary magnetic field H_{0y}. Let a plane shock wave be emitted by this conductor, with its whole surface instantaneously attaining the velocity u toward the field; then an electromagnetic shock wave with velocity c propagates into the field.

Since the field flux between the surface of the ideal conductor and a line at $x = \infty$ is constant, the magnetic field satisfies an equation similar to the equation of conservation of matter:

$$H_0 c = H_1 (c - u).$$

Here $H_1 = H_{0y}$, the field behind the wave. Thus $H_1 = H_0 c/(c - u)$. When H changes on the wave, so does E, and $\Delta E_z = -\Delta H_y$ (where Δ denotes the change at the wave front), so that behind the front $E = -H_0 u/(c - u)$.

We note that in the coordinate system moving with the moving wall, there is no electric field. Indeed, on going from one coordinate system to another, the Lorentz force is conserved, that is

$$\mathbf{E} + [\mathbf{u} \times \mathbf{H}]/c = \mathbf{E}' + [\mathbf{u}' \times \mathbf{H}']/c.$$

In the coordinate system moving with the moving wall (denoted by a prime) $u' = 0$, so that

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}[\mathbf{u} \times \mathbf{H}] = -\mathbf{k}\frac{H_0 u}{c - u} + \frac{\mathbf{k}}{c} u H_1 = 0.$$

Let us further calculate the pressure of the field on the moving wall. The pressure due to the magnetic field H_0 on the wall is $p_0 = H_0^2/8\pi$, so that the pressure on the moving wall is $p_1 = H'^2/8\pi$. On going over from one system of coordinates to another, the quantity $H^2 - E^2$ is conserved, so that $H'^2 - E'^2 = H_1^2 - E_1^2$, and since $E' = 0$, we have

$$p_1 = (H_1^2 - E_1^2)/8\pi.$$

Let us evaluate the thickness of the electromagnetic-field wave front due to a shock wave from a perfect conductor. This thickness does not vanish, since the front of the shock wave moving with a velocity $D \approx 10^6$ cm/sec is spread over a thickness of the order of the lattice constant $a \approx 10^{-8}$ cm of the conductor, or over a time of the order of $\tau = a/D \approx 10^{-14}$ sec. This causes waves spread out in the vacuum over a thickness of the order of $\ell \approx \tau c \approx 3 \times 10^{-4}$ cm, which is extremely small. In actuality the spread is determined by the finite electrical resistance of real conductors. This may be of several orders greater than the above; let us consider it in more detail.

In order to evaluate the thickness of the wave front in the vacuum, let us first consider the wave in the conductor, and then knowing the time characteristic of its spread we can find the thickness of the wave in the vacuum.

Consider a shock wave moving through a real conductor containing a field H_0, this shock wave increasing the density of the material by a factor δ. Let us assume that behind the wave the matter everywhere has the same velocity and density. Let us consider the electromagnetic field accompanying this wave.

In the coordinate system in which **E** is along the z axis, **H** is along the y axis, and **j** is along the z axis (where **j** is the current density), Maxwell's equations with $\epsilon = \mu = 1$ can be written

$$\frac{1}{c}\frac{dE}{dt} = \frac{dH}{dx} - \frac{4\pi}{c} j, \quad \frac{1}{c}\frac{dH}{dt} = \frac{dE}{dx}, \quad j = \lambda\left(E + \frac{uH}{c}\right). \tag{1.1}$$

Here λ is the conductivity, and u is the mass velocity of the substance (in front of the shock wave $u = 0$).

For a steady wave propagating with a velocity D, the quantities H, E, and j depend only on $q = x - Dt$. Then Eq. (1.1) leads to

$$-\frac{D}{c}\frac{dE}{dq} = \frac{dH}{dq} - \frac{4\pi\lambda}{c}\left(E + \frac{uH}{c}\right), \quad -\frac{D}{c}\frac{dH}{dq} = \frac{dE}{dq}. \tag{1.2}$$

From this we obtain

$$d^2E / dq^2 = (-dE / dq)\, 4\pi\lambda (D - u) / (c^2 - D^2).$$

Let us write the solution of this equation in the form

$$E = \begin{cases} A_1 \exp\left\{-\dfrac{4\pi\lambda D}{c^2 - D^2} q\right\} + B_1 & \text{for } q > 0, \\ A_2 \exp\left\} -\dfrac{4\pi\lambda (D-u)}{c^2 - D^2} q\right\} + B_2 & \text{for } q < 0. \end{cases} \tag{1.3}$$

The second of Eqs. (1.2) gives

$$H = \begin{cases} H_1 + \dfrac{c}{D} A_1 \left(1 - \exp\left\{-\dfrac{4\pi\lambda D}{c^2 - D^2} q\right\}\right) & \text{for } q > 0, \\ H_1 + \dfrac{c}{D} A_2 \left(1 - \exp\left\{-\dfrac{4\pi\lambda (D-u)}{c^2 - D^2} q\right\}\right) & \text{for } q < 0. \end{cases} \tag{1.4}$$

Here H_1 is the field on the shock front. It is obvious that H cannot change discontinuously on the front, since this would mean an infinite current density, which cannot occur if λ is finite.

The solution given by (1.3) and (1.4) contains five constants which are determined by the following conditions: in the unperturbed region $H(q = +\infty) = H_0$, and $E(q = +\infty) = 0$; far behind the front $E(q = -\infty) \neq \infty$; on the shock wave E is continuous, or $E(+0) = E(-0)$.

The last condition can be obtained by considering a rectangular circuit of dimension a by b, whose sides of length a are parallel to the wave front. For such a circuit

$$\int_{-\infty}^{+\infty} \left(\oint \mathbf{j}\, d\mathbf{l}\right) dt = \text{const} \int_{-\infty}^{\infty} \frac{d\Phi}{dt} dt = \text{const}\, \Delta\Phi,$$

where Φ is the magnetic flux. But $\oint \mathbf{j}\, d\mathbf{l} = j_1 a - j_2 a$ (where j_1 is the current in the left member, and j_2 is that in the right member of length a of the circuit, and there is no current in the members of length b). But in view of the fact that the process is stationary,

$$\int_{-\infty}^{+\infty} j_1 dt = \int_{-\infty}^{+\infty} j_2\, dt, \text{ i.e. } \Delta\Phi = 0, \quad H_0 ab = a\frac{b}{\delta} H(-\infty)$$

so that finally $H(-\infty) = H_0\delta$; this means that far behind the wave front the magnetic field strength has increased by the same factor as the matter density.

Without going through the simple operations, we present only the final result:

$$H = \begin{cases} H_0[1 + (\delta - 1)e^{-q/l}] & \text{for } q > 0, \\ H_0\delta & \text{for } q < 0, \end{cases} \quad E = \begin{cases} -(D/c)\, H_0 (\delta - 1)\, e^{-q/l} & \text{for } q > 0, \\ -(D/c)\, H_0 (\delta - 1) & \text{for } q < 0, \end{cases} \tag{1.5}$$

$$j = \begin{cases} -(\lambda D/c)\, H_0 (\delta - 1)\, e^{-q/l} & \text{for } q > 0, \\ 0 & \text{for } q < 0, \end{cases} \qquad l = \frac{c^2 - D^2}{4\pi\lambda D}$$

The total current is

$$I = \int_{-\infty}^{+\infty} j\, dq = -H_0 (\delta - 1) \frac{c^2 - D^2}{4\pi c}.$$

As can be seen from (1.5), the perturbation in the magnetic field leads the shock wave by an effective length ℓ or a time $\tau = \ell/D$. On coming to the boundary of the conductor and the vacuum, this wave causes an electromagnetic wave in the vacuum with the same characteristic time of spread, or with a front thickness* $b = \tau c$. For the usual conductors this quantity is much greater than the spread related to that of the shock wave in the matter itself; for instance, in copper (at room temperature $\lambda = 5.8 \times 10^{17}\ \text{sec}^{-1}$) $b = 14.8$ cm (we have taken $D = 5$ km/sec). We note that the thickness ℓ of the wave in the copper itself is $2.5\,\mu$.

The wave front can be decreased greatly by cooling the conductor to decrease its resistance. For instance, when copper is cooled to -253°C, its resistance is decreased by a factor of 170, and the spread of the wave in the vacuum decreases to 14.8 cm/170 = 0.9 mm.

We note that the total current I accompanying the shock wave in the conductor is independent of the conductivity, and that behind the shock front the current vanishes, so that the conductivity plays no role behind the shock wave. Thus if one is able to achieve a narrow front by increasing the conductivity by cooling, this result will not be affected even if the high conductivity is destroyed by the shock wave itself (for instance, because of heating).

In the following sections we shall consider a converging cylindrical electromagnetic-field wave, assuming it to be a mathematically rigorous discontinuity.

2. CONVERGING CYLINDRICAL FIELD WAVE

In principle a converging wave can be obtained by suddenly turning on a circular current simultaneously over the whole surface of a cylinder, by reflecting a plane or diverging wave from a curved mirror, or by moving a cylindrical conducting surface in a magnetic field towards or away from its axis.

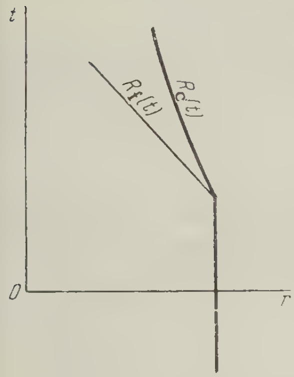

FIG. 1. The formation of a converging cylindrical electromagnetic-field shock wave. The vacuum lies to the left of the heavy line.

In the case of cylindrical symmetry, Maxwell's equations (H is parallel to the x axis, and E is circular, i.e., perpendicular to r and x) become

$$\frac{1}{c}\frac{\partial E}{\partial t} = -\frac{\partial H}{\partial r}, \quad \frac{1}{c}\frac{\partial H}{\partial t} = -\frac{1}{r}\frac{\partial (rE)}{\partial r}. \tag{2.1}$$

Eliminating H or E from these equations, we obtain, respectively,

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{1}{\partial r}\left[\frac{1}{r}\frac{\partial (rE)}{\partial r}\right] = 0, \tag{2.2}$$

$$\frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial H}{\partial r}\right] = 0. \tag{2.3}$$

The equation for H is simply the wave equation for the cylindrical case. That for E is not the wave equation.

Let us consider the variation of the cylindrical wave amplitude as it moves towards the axis, first treating a wave within a cylindrical cavity of radius R_c in an ideal conductor. The motion of the conductor and the wave in this case are shown in Fig. 1.

The total flux of H in the cavity is conserved, so that

$$\Phi = 2\pi \int_0^{R_c} rH\, dr = \text{const.}$$

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* Here we have obtained only the width in the vacuum corresponding to the field perturbation in front of the shock wave. The amplitude of this part of the wave and its profile can be found only by the complete solution of the problem of the wave leaving the conductor.

Differentiating this identity twice with respect to time, using (2.3), and $dR_f/dt = -c$, and $dR_c/dt = U$ (where R_f is the radius of the wave front), gives

$$c\left[2R_f\frac{d(H_f-H_0)}{dt}-c(H_f-H_0)\right]+c^2R_c\left(\frac{\partial H}{\partial r}\right)_c+\frac{d(H_cR_cU)}{dt}=0.$$

The second group of terms, which refers to the boundary, vanishes. This is easily seen by considering a field due not to a sharp shock wave, but to one which is somewhat spread out. Then the first group of terms vanishes, and the second, which contains only quantities referring to the boundary, does not change.* Therefore the first group also vanishes, which leads to

$$H_f-H_0=\text{const}\cdot R_f^{-1/2}. \tag{2.4}$$

Thus the amplitude of the converging cylindrical wave increases without bound as $R_f^{-1/2}$ in its approach toward the axis. We have obtained this result for the special case of a wave within a cavity in an ideal conductor, although it is valid for any cylindrical shock wave, since waves from the surface of the cylinder do not catch up to the shock wave; therefore the behavior of the wave amplitude on the front is determined entirely by its magnitude (that is the initial amplitude at the point at which the wave is formed) and is independent of other field changes on the boundary of the cylinder.

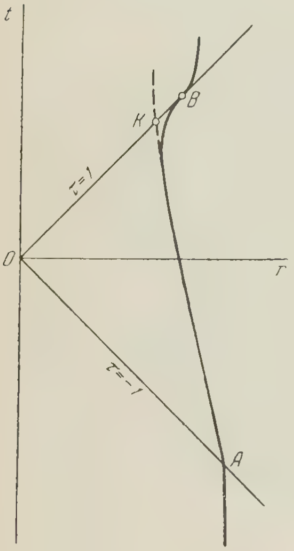

FIG. 2.

On the cylindrical wave front $E_f - E_0$ also changes according to

$$E_f-E_0=\text{const}\ R_f^{-1/2}. \tag{2.5}$$

Equations (2.4) and (2.5) are not approximations valid only for large or small amplitudes; they describe the behavior of a field with the same accuracy as do Maxwell's equations.

For a complete description of H and E behind the shock wave we must solve (2.2) and (2.3).

Henceforth we shall consider only the particular solution corresponding to a converging wave, which describes the limiting behavior of a field close to the axis and close to the time of focusing, which means that we shall solve the self-similar problem.

3. SELF-SIMILAR SOLUTION FOR A CONVERGING CYLINDRICAL WAVE

Since we know how the wave amplitude varies, we may immediately assert that the self-similar solution is of the form

$$E=\mathcal{E}_0\sqrt{\frac{r_0}{r}}\,e(\tau),\qquad H=H_0+\mathcal{H}_0\sqrt{\frac{r_0}{r}}\,h(\tau), \tag{3.1}$$

where $\tau = ct/r$. Here t is the time calculated from the instant of focusing (before focusing, $t < 0$), and $\mathcal{E}_0$ and $\mathcal{H}_0$ are the amplitudes of the electric and magnetic fields on the wave front at the time when the wave front is at a distance r_0 from the axis. For a wave moving towards the axis, $\mathcal{H}_0 = -\mathcal{E}_0$. On a converging wave front, $\tau = -1$, so that

$$e(-1)=1,\quad h(-1)=1. \tag{3.2}$$

A diagram of the phenomenon in the r, t plane is shown in Fig. 2.

Inserting (3.1) into Maxwell's equations (2.2) and (2.3), we obtain equations for the new functions

$$e''(1-\tau^2)-2\tau e'+3e/4=0,\quad h''(1-\tau^2)-2\tau h'-h/4=0. \tag{3.3}$$

We solve these equations by means of a power series in $(1+\tau)$ with the conditions (3.2) for the interval $-1 \le \tau \le 1$ (between the incident and reflected waves), obtaining

$$e(\tau)=1-\sum_{n=1}^{\infty}\frac{(2n+1)!!\,(2n-3)!!}{2^{3n}(n!)^2}(1+\tau)^n, \tag{3.4}$$

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*This reasoning is due to B. P. Mordvinov.

$$h(\tau)=\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{n!}\right]^2\frac{(1+\tau)^n}{2^{3n}} \tag{3.5}$$

(for $n = 0$ we take $(2n-1)!! = +1$).

These series converge for $\tau < 1$, and diverge for $\tau \geq 1$, so that as a result of reflection from the axis there arises a wave whose amplitude is unbounded not only at the cylindrical axis, but also at a finite distance from it.

Such a cumulation is a qualitatively new phenomenon. In converging waves previously considered, unbounded amplitudes arose only for small times and in small regions of space (at the focus). In our case, however, unbounded amplitudes arise at all points of space, not only at the cylindrical axis.

We note that nowhere in the solution have we assumed that we are dealing with a compression wave, so that all the results obtained refer also to a rarefaction wave, which can be formed, for instance, by the sudden expansion of a cavity in a conductor. In this case the field strength on the wave front $H_f = H_0 - \Delta H$ will decrease as the amplitude ΔH increases, vanishing at some distance r and thereafter approaching $-\infty$.

We wish, further, to find a solution of the equations for e and h in the interval $1 < \tau < \infty$, that is behind the wave reflected from the axis. We write this solution in the form of a power series in $1/\tau$ with the condition $e = h = 0$ when $\tau = \infty$ (that is with the condition $E(r=0) \not\equiv \infty$ and $H(r=0) \not\equiv \infty$), obtaining

$$e=\frac{a_0}{\tau^{3/2}}\sum_{n=0}^{\infty}\frac{(4n+1)!!}{2^{4n}n!(n+1)!}\frac{1}{\tau^{2n}},\quad h=\frac{b_0}{\tau^{1/2}}\left[1+\sum_{n=1}^{\infty}\frac{(4n-1)!!}{2^{4n}(n!)^2}\frac{1}{\tau^{2n}}\right]. \tag{3.6}$$

These series converge for $\tau > 1$, but diverge for $\tau \leq 1$, so that close to the reflected wave and behind its front E and H increase without limit.

The as yet unknown quantities a_0 and b_0 can be obtained from the following two conditions: first, from Maxwell's equations, and second, from the condition that $e + h$ are conserved on passing through the wave front reflected from the axis (see below). Let us consider each condition separately.

Inserting E and H in their self-modeling form (3.1) into the second of Maxwell's equations (2.1), we obtain

$$dh/d\tau+\tau\, de/d\tau-e/2=0.$$

Inserting the expressions for $de/d\tau$, $dh/d\tau$, and e behind the wave into this expression, we obtain

$$b_0=-4a_0. \tag{3.7}$$

Let us now consider the second condition. On the front of a diverging wave, $\Delta E = \Delta H$, or $E_2 - H_2 = E_1 - H_1 = \text{const}$. In our case

$$E-H=\mathcal{E}_0\sqrt{\frac{r_0}{r}}e-\mathcal{H}_0\sqrt{r_0/r}\,h-H_0=\mathcal{E}_0\sqrt{r_0/r}\,(e+h)-H_0=\text{const},$$

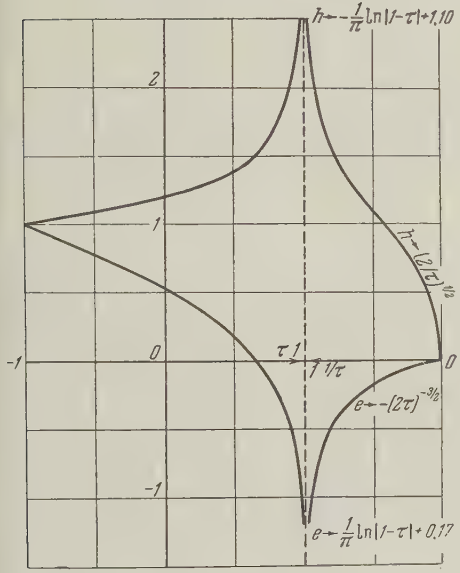

FIG. 3. The self-similar solution for a converging cylindrical electromagnetic-field wave: $H = H_0 - \text{const} \times h/\sqrt{r}$, $E = \text{const} \times e/\sqrt{r}$.

so that $e + h = \text{const}$. Using (3.4) and (3.5), we find that in front of the wave

$$e+h=2\left[1-\sum_{n=1}^{\infty}\frac{(2n-1)!!(2n-3)!!}{2^{2n}(n!)^2}\right]. \tag{3.8}$$

This series converges, so that the invariant $e + h$ has a finite value on the wave front.

Using the solution (3.6) for the region behind the front, as well as relation (3.7), we find that in that region

$$e+h=-3a_0\left[1+\sum_{n=1}^{\infty}\frac{(4n-1)!!}{2^{4n}n!(n+1)!}\right]. \tag{3.9}$$

Comparing (3.8) and (3.9) we obtain a_0 in the form of the ratio of two series; numerical calculation gives $a_0 = 0.3536$. It would seem that the accurate solution is

$$a_0 = -1/2\sqrt{2}; \text{ so that } b_0 = \sqrt{2}.$$

Let us consider in more detail the divergence of e and h close to the reflected wave front. Introducing the new variables $z = \tau - 1$ and $y = e/e'$ into Eq. (3.2) for e, we are led to

$$(1-y')\,y^{-2}z(2+z) + 2(1+z)/y - {}^3/_4 = 0.$$

When $|z| \ll 1$ and $|y| \ll 1$, this equation can be simplified, becoming $dy/dz = 1 + y/z$. We then have

$$y = z(\ln z + \text{const}) = e/e'$$

and finally

$$e = A_0 + A_1 \ln|1-\tau|. \qquad (3.10)$$

This equation describes the asymptotic behavior in the neighborhood of the diverging wave front on both its sides.

Let us determine the coefficient of the logarithm in this expression. We shall choose A_0 and A_1 such that the difference between the exact expression e_{ex} as given by (3.6) and the asymptotic one e_a as given by (3.10) vanish for $\tau = 1$. Let us write e_{ex} and e_a in the form of power series in $(1+\tau)$:

$$e_{ex} = 1 + \sum_{n=1}^{\infty} a_n (1+\tau)^n, \quad e_a = A_0 + A_1 \ln 2\left(1 - \frac{1+\tau}{2}\right) = A_0 + A_1 \ln 2 - A_1 \sum_{n=1}^{\infty} \frac{(1+\tau)^n}{2^n n},$$

$$e_{ex} - e_a = 1 - A_0 - A_1 \ln 2 + \sum_{n=1}^{\infty} (a_n + A_1/2^n n)(1+\tau)^n.$$

The series in the last expression should converge when $\tau = 1$. From this it follows that

$$\lim_{n\to\infty} (1 + a_n n 2^n / A_1) = 0,$$

since otherwise the series diverges as $\Sigma\, 1/n$. Therefore

$$A_1 = -\lim_{n\to\infty} 2^n n a_n = \lim_{n\to\infty} \frac{(2n+1)!!\,(2n-3)!!}{2^{2n} n!\,(n-1)!} = \frac{1}{\pi}$$

(in order to calculate the limit, it is convenient to express the factorials by Stirling's formula).

The coefficient of $\ln|1-\tau|$ in the asymptotic expression for e behind the front is found in the same way, and is $A_2 = -a_0 2\sqrt{2}/\pi$. With the exact equation $a_0 = -\frac{1}{2}\sqrt{2}$, we have $A_2 = A_1 = 1/\pi$, which means that the coefficient of the logarithm in the asymptotic expressions for e on both sides of the wave front are the same. It is clear that the coefficient in the asymptotic expression for h differs from this only in its sign (e + h is finite on the front).

Using the same method to calculate A_0 and the analogous quantity for h, we obtain asymptotic expressions for e and h which are found to be of the same form on both sides of the waves, and are given by

$$e = \frac{1}{\pi}\ln|1-\tau| + 0.17, \quad h = -\frac{1}{\pi}\ln|1-\tau| + 1.10. \qquad (3.11)$$

Close to the cylindrical axis after focusing,

$$e \to -1/2\sqrt{2}\,\tau^{3/2} = -(r/2ct)^{3/2}, \quad h \to (2/\tau)^{1/2} = \sqrt{2r/ct}.$$

From this it is seen that the electric field close to the axis is

$$E = -\mathscr{E}_0 \sqrt{r_0/r}\,(r/2ct)^{3/2} = -\mathscr{E}_0 r_0^{1/2} r/(2ct)^{3/2},$$

so that $E \equiv 0$ on the axis. The magnetic field is

$$H = H_0 + \mathscr{H}_0 \sqrt{2r_0/ct},$$

so that on the axis $H - H_0$ decreases as $t^{-1/2}$ after focusing.

The fundamental quantitative results of the self-similar solution are shown in Fig. 3, which gives graphs of $e(\tau)$ and $h(\tau)$ calculated from the formulas for the exact solution. The same figure gives the asymptotic formulas describing e and h in the neighborhood of the reflected wave and the axis.

4. FURTHER REMARKS

(a) Region of Applicability of the Self-Similar Solution

As can be seen from the solution itself, it is applicable for waves of arbitrary amplitude, not only for strong waves as in gas dynamics. The solution gives an exact description of all the phenomena in a finite (r,t) region if, for instance, the surface of an ideal conductor surrounding a cylindrical cavity moves with $r(t)$ corresponding to the solution obtained. If, however, this surface is suddenly caused to move and thereafter moves in an arbitrary way, the self-similar solution is not valid in the whole (r,t) region, but describes only the behavior close to the axis at the moment of focusing.

(b) Reflection of a Diverging Wave from a Conductor

It is not difficult to see that the radius $r(t)$ of a cylinder within which the total flux Φ is conserved has, in the case of the self-similar solution and a compression wave, the form of the $r(t)$ curve shown in Fig. 2: in view of the unbounded values of H on the diverging wave, this curve should be tangent to the line $r = ct$, which means that the conducting surface should move away from the axis with the velocity of light c (behind the front the velocity decreases again).

If before $r = ct$ the conductor does not start moving sharply in the other direction, but moves as shown by the dotted line, then a wave of infinite amplitude will be reflected from it at point K. We have not considered the further behavior of such a wave as it converges, although it may be of interest.

(c) Cylindrical Acoustical Wave

Let us consider a weak converging cylindrical shock wave for which $u \ll c$, where c is the velocity of sound. The solution of this problem reduces to solving the wave equation for the pressure

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = 0.$$

This equation for p is exactly the same as (2.5) for H in the electromagnetic wave problem. Therefore their self-similar solutions are also identical, and on the wave front reflected from the axis the pressure is also infinite. Thus this qualitative property of cumulation is not peculiar to the electromagnetic nature of the waves, but is a property of converging cylindrical waves described by the wave equation.

We note that for a spherical acoustical wave one may obtain a solution without the assumption that it is self-similar, and this makes it possible to verify the assertion that the self-similar solution describes the limiting form of the solution at the center close to the focus. A test shows that the self-similar solution for a sphere is indeed identical with the limiting form of the general solution, although in this case no singularities occur on the wave reflected from the center. (The amplitudes of the incident and reflected waves are equal, but opposite in sign; between the waves the pressure is maintained at every point.)

In conclusion we thank V. P. Mordvinov, D. G. Lominadze, A. B. Govorkov, and A. A. Bunatian, with whom we have had extremely valuable discussions of the material contained in the present article.

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A QUANTUM MECHANICAL TRANSPORT EQUATION FOR ELECTRONS IN METALS

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We derive a quantum mechanical transport equation for electrons in a metal in the field of an electromagnetic wave. In contrast to the usual treatment, our equation is not based on the assumption that the photon energy is small compared to kT.

DURING the last decade there has been developed a theory of the skin-effect in metals* based on the following transport equation for the electron distribution function $f(\mathbf{p}, \mathbf{r}, t)$

$$\partial f/\partial t + (\mathbf{p}/m)\,\partial f/\partial \mathbf{r} + e\mathbf{E}\,\partial f/\partial \mathbf{p} = I(f), \tag{1}$$

where $I(f)$ is the collision integral. In this equation the electrical field $\mathbf{E}$ plays the role of a classical force. However, if the frequency ω of the electromagnetic field is sufficiently high, one must take quantum effects into account, even if the threshold for the internal photoeffect, i.e., for a transition of an electron into another band, is not yet reached. From the quantum mechanical point of view the interaction of an electromagnetic wave with the electrons takes place through the absorption and emission of photons (with a simultaneous absorption and emission of phonons in the lattice). If in these processes the change in the electron energy $\hbar\omega$ is small compared to the breadth of the tail of the Fermi distribution kT, one can consider the gain of energy by the electron to be practically continuous.

As long as the condition

$$\hbar\omega \ll kT \tag{2}$$

is satisfied, one can thus use classical methods and use the transport equation (1). We notice that condition (2) is extremely far reaching. At room temperatures ($T \cong 300°$) it involves $\lambda \gg 40\mu$ and for $T = 10°$ we have $\lambda \gg 0.2$ cm, which means that quantum effects can be of importance even in the radio range.

If $\hbar\omega \gtrsim kT$, it is necessary to treat the problem quantum mechanically. This is, in particular, true for the infrared region of the spectrum, where usually inequality (2) holds in reverse. Although this fact has been known for a long time (see, for instance, Sec. 53 of Ref. 2) up to the present nobody has carried out anything like the complete investigation of this case. The only attention paid to this problem was in a paper in 1954.[3] Holstein has calculated the volume absorption in a metal using perturbation theory. In this case he assumes that the electron distribution and the field do not depend on the position coordinates. Holstein's results are therefore, generally speaking, only applicable in the case of the normal skin-effect. On the other hand, if we exclude the region of very high temperatures, the skin-effect in the infrared region of the spectrum is usually anomalous. It is thus necessary to construct a method which enables us to consider simultaneously quantum effects and the anomalous character of the skin-effect. The present paper is devoted to obtaining such an equation.

For the time being, to fix our ideas, we shall consider a system of N electrons in a macroscopic volume V.

Such a system can in quantum mechanics conveniently be described through a density operator $\hat{F}$ which satisfies the equation[4]

$$\partial \hat{F}/\partial t = [\hat{H}, \hat{F}] \equiv (1/i\hbar)(\hat{H}\hat{F} - \hat{F}\hat{H}) \tag{3}$$

and the normalization condition $\mathrm{Tr}\,\hat{F} = 1$; in Eq. (3) $\hat{H}$ is the Hamiltonian of the system.

The operator $\hat{F}$ gives a complete quantum mechanical description of the system since the average value of any dynamical variable R can be calculated from the formula

$$R_{\mathrm{Av}} = \mathrm{Sp}\,(\hat{R}\hat{F}). \tag{4}$$

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*See, for instance, the survey article by Ginzburg and Motulevich.[1]

In quantum mechanics one normally uses the density matrix $\rho(x,x')$ which is the x-representation of the operator $\hat{F}$.

We shall use the so-called mixed representation of the density matrix which is determined in the following way[5]

$$f(p, x, t) = (2\pi\hbar)^{-3N}\int d\tau\rho\left(x - \frac{\tau}{2},\ x + \frac{\tau}{2}\right)e^{-i\tau p/\hbar}.$$

The "quantum distribution function" which is introduced in this way does not have the meaning of a probability density for the state with coordinates p and x, as can be seen, in particular, from the fact that the function f(p, x, t), as follows from its definition, though essentially positive is not always so without fail. However, if we use it, we obtain the most complete analogy with the classical case. In particular, Eq. (4) for the averages now becomes

$$\langle\hat{R}(\hat{p}, \hat{x})\rangle_{\mathrm{Av}} = \int dp\, dx\, R(p, x) f(p, x, t).$$

For the sake of simplicity we shall consider one-particle distribution functions. If an external electromagnetic field is present this function can be determined as follows

$$f(\mathbf{P}, \mathbf{r}, t) = (2\pi\hbar)^{-3}\int d\tau\rho\left(\mathbf{r} - \frac{\tau}{2},\ \mathbf{r} + \frac{\tau}{2}\right)\exp\left\{-\frac{i}{\hbar}\tau\left(\mathbf{P} + \frac{e}{c}\mathbf{A}\right)\right\},$$

where $\mathbf{P}$ is the generalized momentum and $\mathbf{A}$ the vector potential of the field. For physical reasons it is convenient, however, to deal with distribution functions of the ordinary momenta. We change at the same time the normalization of the distribution function in such a way that it gives the average relative occupation number with respect to the momenta. We make thus the following substitution

$$N(2\pi\hbar)^3 f(\mathbf{P}, \mathbf{r}, t) = f'(\mathbf{p}, \mathbf{r}, t); \quad \mathbf{P} = \mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r}, t). \tag{5}$$

Since the time is involved in the transformation we have

$$N(2\pi\hbar)^3\frac{\partial}{\partial t}f(\mathbf{P}, \mathbf{r}, t) = \frac{\partial}{\partial t}f'(\mathbf{p}, \mathbf{r}, t) + \frac{e}{c}\mathbf{A}(\mathbf{r}, t)\frac{\partial}{\partial\mathbf{p}}f'(\mathbf{p}, \mathbf{r}, t).$$

The primes will be omitted in the following.

It can easily be shown that the electrical current density can now be written in the usual way

$$\mathbf{j}(\mathbf{r}, t) = \frac{2e}{(2\pi\hbar)^3}\int\frac{\mathbf{p}}{m}f(\mathbf{p}, \mathbf{r}, t)\, d\mathbf{p}.$$

Finally one obtains easily the relation

$$F_{\mathbf{p},\mathbf{p}'} = \frac{1}{N(2\pi\hbar)^3}\int d\mathbf{r}\, f\left(\frac{\mathbf{p}+\mathbf{p}'}{2}, \mathbf{r}, t\right)e^{i\mathbf{r}(\mathbf{p}-\mathbf{p}')/\hbar}. \tag{6}$$

We consider a system consisting of electrons, photons and lattice phonons. We shall use methods obtained from the transport equations developed by Bogoliubov and Gurov.[6,7]

The conclusions below are based on perturbation theory; this means that we assume the interaction of the electrons with the lattice vibrations and the electromagnetic field to be small. In metal optics the occupation numbers of photons are always large compared to unity. The electromagnetic field can thus be considered to be an external, classical field, satisfying Maxwell's equations. Under our assumptions, the density operator of the system $\hat{F}$ will depend on the coordinates of N electrons $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$, on the momenta of S phonons $(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_S)$, and on the time t.

We write the Hamiltonian of the system in the form

$$\hat{H}(1, \ldots N; \mathbf{q}_1, \ldots, \mathbf{q}_S; t) = \sum_{k=1}^{N}\hat{H}(k) + \sum_{k=1}^{S}H(\mathbf{q}_k) + \sum_{i=1}^{N}\sum_{k=1}^{S}\hat{U}(i, k);$$

$$\hat{H}(k) = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial\mathbf{r}_k} + \frac{e}{c}\mathbf{A}(\mathbf{r}_k, t)\right)^2,\ \hat{H}(\mathbf{q}) = \frac{1}{2}\hbar\nu_q(\hat{b}_{\mathbf{q}}^{+}\hat{b}_{\mathbf{q}} + \hat{b}_{\mathbf{q}}\hat{b}_{\mathbf{q}}^{+}),$$

where $\hat{b}_{\mathbf{q}}^{+}$ and $\hat{b}_{\mathbf{q}}$ are the creation and annihilation operators of phonons of momentum $\mathbf{q}$,

$$(N_{\mathbf{q}} + 1\,|\,\hat{b}_{\mathbf{q}}^{+}\,|\,N_{\mathbf{q}}) = \sqrt{N_{\mathbf{q}} + 1}, \quad (N_{\mathbf{q}} - 1\,|\,\hat{b}_{\mathbf{q}}\,|\,N_{\mathbf{q}}) = \sqrt{N_{\mathbf{q}}}.$$

Finally, the operator of the interaction between electrons and phonons is of the form,

$$\hat{U}(1, \mathbf{q}) = C(\Delta |\mathbf{q}| / V 2Mu)^{1/2} (\hat{b}_{\mathbf{q}}^{+} e^{-i\mathbf{qr}/\hbar} + \hat{b}_{\mathbf{q}} e^{i\mathbf{qr}/\hbar}),$$

where C is a constant of the order of magnitude of the average electron energy in the metal,[8] M the mass of an atom in the lattice, u the velocity of sound, and Δ the volume of an elementary cell in the crystal.

We introduce now operators that depend only on a fraction of all the particles, defined as follows

$$\hat{F}^{\nu,\sigma}(1 \ldots, \nu; \mathbf{q}_1, \ldots \mathbf{q}_\sigma; t) = \operatorname*{Sp}_{\binom{\nu+1, \ldots N}{\sigma+1, \ldots, S}} \hat{F}^{N,S}(1, \ldots N; \mathbf{q}_1, \ldots \mathbf{q}_S; t).$$

Applying this operation to both sides of equation (3) we find

$$\frac{\partial}{\partial t}\hat{F}^{\nu\sigma} = \left[\sum_{k=1}^{\nu} \hat{H}(k) + \sum_{k=1}^{\sigma} \hat{H}(\mathbf{q}_k), \hat{F}^{\nu\sigma}\right] + \sum_{i=1}^{\nu}\sum_{k=1}^{\sigma} [\hat{U}(i, \mathbf{q}_k), \hat{F}^{\nu\sigma}]$$

$$+ (N - \nu) \sum_{k=1}^{\sigma} \operatorname*{Sp}_{(\nu+1)} [\hat{U}(\nu+1, \mathbf{q}_k) \hat{F}^{\nu+1,\sigma}] + \sum_{i=1}^{\nu} \sum_{k=\sigma+1}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(i, \mathbf{q}_k) \hat{F}^{\nu,\sigma+1}].$$

Hence it follows in particular $(N \gg 1)$

$$\frac{\partial \hat{F}(1)}{\partial t} = [\hat{H}(1), \hat{F}(1)] + \sum_{k=1}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(1, \mathbf{q}_k), \hat{F}(1, \mathbf{q}_k)], \quad \frac{\partial \hat{F}(\mathbf{q})}{\partial t} = [\hat{H}(\mathbf{q}), \hat{F}(\mathbf{q})] + N \operatorname*{Sp}_{(1)} [\hat{U}(1, \mathbf{q}), \hat{F}(1, \mathbf{q})],$$

$$\frac{\partial \hat{F}(1, \mathbf{q})}{\partial t} = [\hat{H}(1) + \hat{H}(\mathbf{q}) + \hat{U}(1, \mathbf{q}), \hat{F}(1, \mathbf{q})] + N \operatorname*{Sp}_{(2)} [\hat{U}(2, \mathbf{q}) \hat{F}(1, 2; \mathbf{q})] + \sum_{k=2}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(1, \mathbf{q}_k), \hat{F}(1; \mathbf{q}, \mathbf{q}_k)]. \tag{7}$$

If there is no interaction operator, the dependence on the coordinates of several particles is obviously of the form of a product of one-particle operators; in particular,

$$\hat{F}(1, \mathbf{q}) = \hat{F}(1) \hat{F}(\mathbf{q}), \quad \hat{F}(1, 2) = (1 - \hat{P}_{12}) \hat{F}(1) \hat{F}(2),$$
$$\hat{F}(1, 2; \mathbf{q}) = \hat{F}(\mathbf{q}) (1 - \hat{P}_{12}) \hat{F}(1) \hat{F}(2), \quad \hat{F}(1; \mathbf{q}, \mathbf{q}') = \hat{F}(1) \hat{F}(\mathbf{q}) \hat{F}(\mathbf{q}'),$$

where $\hat{P}_{12}$ is the "permutation" operator.* If the interaction is not zero we introduce instead of $\hat{F}(1, \mathbf{q})$ the correlation operator

$$\hat{G}(1, \mathbf{q}) = \hat{F}(1, \mathbf{q}) - \hat{F}(1) \hat{F}(\mathbf{q}).$$

Since we have assumed the interaction between electrons and phonons to be small, the operator $\hat{G}$ is obviously also small. For problems connected with the behavior of metals in an electromagnetic field (in particular, for calculating the current) it is sufficient to restrict ourselves in the transport equation to terms quadratic in the magnitude of the interaction between electrons and phonons. Correspondingly one can retain in Eq. (7) only the terms linear in this quantity. In the same approximation we can take for the state operator of the phonons its equilibrium value

$$(N_{\mathbf{q}} | \hat{F}(\mathbf{q}) | N'_{\mathbf{q}}) = \delta(N_{\mathbf{q}}, N'_{\mathbf{q}}) \, \delta(N_{\mathbf{q}}, \overline{N}_{\mathbf{q}}), \quad \overline{N}_{\mathbf{q}} = [e^{h\nu_{\mathbf{q}}/kT} - 1]^{-1}, \tag{8}$$

where $h\nu_{\mathbf{q}} = u|\mathbf{q}|$ is the energy of a phonon of momentum **q**. If, at the same time, we take into account the fact that the operator $\hat{U}(1, \mathbf{q})$ is non-diagonal in the occupation numbers of the phonons we obtain easily the following set of equations

$$\partial \hat{F}(1) / \partial t = [\hat{H}(1), \hat{F}(1)] + \sum_{k=1}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(1, \mathbf{q}_k), \hat{G}(1, \mathbf{q}_k)] \tag{9}$$

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*The operator $\hat{F}$ is symmetrical with respect to the permutation of identical particles. However, in order to calculate matrix elements we have to deal with antisymmetric combinations of one-electron wave functions, which is inconvenient. It is better to use certain methods to antisymmetrize the operator $\hat{F}$ after which one can take the matrix elements with respect to a product of one-electron ψ-functions. The meaning of the operation P_{12} is clear from the equation

$$(\mathbf{p}_1, \mathbf{p}_2 | \hat{P}_{12} \hat{F}(1) \hat{F}(2) | \mathbf{p}'_1, \mathbf{p}'_2) = F_{\mathbf{p}_1, \mathbf{p}'_2} F_{\mathbf{p}_2 \mathbf{p}'_1}.$$

$$\partial \hat{G}(1, \mathbf{q})/\partial t = [\hat{H}(1) + \hat{H}(\mathbf{q}),\ \hat{G}] + [\hat{U}(1, \mathbf{q}), \hat{F}(1)\hat{F}(\mathbf{q})] - N \underset{(2)}{\mathrm{Sp}}\, [\hat{U}(2, \mathbf{q}), \hat{P}_{12}\hat{F}(1)\hat{F}(2)\hat{F}(\mathbf{q})], \tag{10}$$

where $\hat{\mathrm{F}}(\mathbf{q})$ is of the form (8).

We go now over to an expansion in terms of the magnitude of the vector potential of the electromagnetic field which we write in the form

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r})\, e^{i\omega t} + \mathbf{A}_0^{*}(\mathbf{r})\, e^{-i\omega t}$$

If there is no field the electron distribution is homogeneous and does not depend on the time; any deviation from this distribution will be assumed to be due to the electromagnetic field. It is therefore natural to represent the quantities we are trying to determine in the following form

$$\hat{F}(1, t) = \hat{F}_0(1) + \hat{F}_1(1, t),\ \hat{G}(1, \mathbf{q}, t) = \hat{G}_0(1, \mathbf{q}) + \hat{G}_1(1, \mathbf{q}, t). \tag{11}$$

In these equations the operators $\hat{\mathrm{F}}_0$ and $\hat{\mathrm{G}}_0$ do not depend on the time and describe the equilibrium state of the system of electrons which are interacting with the phonons; however, the operators $\hat{\mathrm{F}}_1$ and $\hat{\mathrm{G}}_1$ which represent the influence of the electromagnetic field will be proportional to the amplitude of this field, A_0, and be of the form

$$\hat{F}_1(1, t) = \hat{F}_{1\omega}(1)\, e^{i\omega t} + \hat{F}_{1\omega}^{+}(1)\, e^{-i\omega t},\quad \hat{G}_1(1, \mathbf{q}, t) = \hat{G}_{1\omega}(1, \mathbf{q})\, e^{i\omega t} + \hat{G}_{1\omega}^{+}(1, \mathbf{q})\, e^{-i\omega t}.$$

Using the notation

$$\hat{B}(1, \mathbf{q}, t) = [\hat{U}(1, \mathbf{q}), \hat{F}(1)\hat{F}(\mathbf{q})] - N \underset{(2)}{\mathrm{Sp}}\, [\hat{U}(2, \mathbf{q}), \hat{P}_{12}\hat{F}(1)\hat{F}(2)\hat{F}(\mathbf{q})],$$

we can write Eq. (10) in the form

$$\partial\hat{G}/\partial t - [\hat{H}(1) + \hat{H}(\mathbf{q}), \hat{G}] = \hat{B}. \tag{12}$$

In the same way as in Refs. 6 and 7 we impose on the solution of this equation the condition that the correlation vanish when the particles go to infinity,

$$\lim_{\tau \to \infty} \hat{S}_{t+\tau}\hat{G}(t+\tau)\,\hat{S}_{t+\tau}^{+} = 0,\ \hat{S}_t = \exp\left\{-\frac{1}{i\hbar}\int_0^t dt'\, [\hat{H}(1, t') + \hat{H}(\mathbf{q})]\right\}. \tag{13}$$

Carrying out the differentiation, we have by virtue of (12)

$$\frac{\partial}{\partial\tau}\hat{S}_{t+\tau}\hat{G}(t+\tau)\,\hat{S}_{t+\tau}^{+} = \hat{S}_{t+\tau}\left\{\frac{\partial}{\partial\tau}\hat{G}(t+\tau) - [\hat{H}(1, t+\tau) + \hat{H}(\mathbf{q}),\ \hat{G}(t+\tau)]\right\}\ \hat{S}_{t+\tau}^{+} = \hat{S}_{t+\tau}\hat{B}(t+\tau)\,\hat{S}_{t+\tau}^{+}.$$

Hence we have after integration

$$\hat{S}_{t+\tau}\hat{G}(t+\tau)\,\hat{S}_{t+\tau}^{+} - \hat{S}_t\hat{G}(t)\,\hat{S}_t^{+} = \int_0^{\tau} d\tau'\hat{S}_{t+\tau'}\hat{B}(t+\tau')\,\hat{S}_{t+\tau'}^{+}.$$

We now go to the limit $\tau \to \infty$; then, by virtue of (13) the first term vanishes and we obtain the solution of (12) in the form

$$\hat{G}(t) = -\int_0^{\infty} d\tau \exp\left\{-\frac{1}{i\hbar}\int_t^{t+\tau} dt'\, [\hat{H}(1,t') + \hat{H}(\mathbf{q})]\right\}\hat{B}(t+\tau)\exp\left\{+\frac{1}{i\hbar}\int_t^{t+\tau} dt'\, [\hat{H}(1,t') + \hat{H}(\mathbf{q})]\right\}.$$

If we restrict ourselves to terms linear in the field we find thus after some simple calculations

$$(\mathbf{P}, N_{\mathbf{q}}|\hat{G}_0|\mathbf{P}', N_{\mathbf{q}}') = -2\pi\hbar\delta_{+}(E_P - E_{P'} + (N_{\mathbf{q}} - N_{\mathbf{q}}')\,h\nu_q)\,(\mathbf{P}, N_{\mathbf{q}}|\hat{B}_0|\mathbf{P}', N_{\mathrm{q}}'),$$

$$(\mathbf{P}, N_{\mathbf{q}}|\hat{G}_{1\omega}|\mathbf{P}', N_{\mathbf{q}}) = -2\pi\hbar\delta_{+}(E_P - E_{P'} + (N_{\mathbf{q}} - N_{\mathbf{q}}')\,h\nu_q + \hbar\omega)$$
$$\times(\mathbf{P}, N_{\mathbf{q}}|\hat{B}_{1\omega}|\mathbf{P}', N_{\mathbf{q}}') - 2\pi\hbar\frac{e}{mc\hbar\omega}(A_0.\,\mathbf{P} - \mathbf{P}')\,[\delta_{+}(E_P - E_{P'}$$
$$+ (N_{\mathbf{q}} - N_{\mathbf{q}}')\,h\nu_q + \hbar\omega) - \delta_{+}(E_P - E_{P'} + (N_{\mathbf{q}} - N_{\mathbf{q}}')\,h\nu_q)](\mathbf{P}, N_{\mathbf{q}}|\hat{B}_0|\mathbf{P}', N_{\mathrm{q}}'),$$

where

$$\hat{B}_0(1, \mathbf{q}) = [\hat{U}(1, \mathbf{q}),\ \hat{F}_0(1)\hat{F}(\mathbf{q})] - N\underset{(2)}{\mathrm{Sp}}\, [\hat{U}(2, \mathbf{q}), \hat{P}_{12}\hat{F}_0(1)\hat{F}_0(2)\hat{F}(\mathbf{q})],$$

$$\hat{B}_{1\omega}(1, \mathbf{q}) = [\hat{U}(1, \mathbf{q}), \hat{F}_{1\omega}(1)\hat{F}(\mathbf{q})] - N\underset{(2)}{\mathrm{Sp}}\, [\hat{U}(2, \mathbf{q}), \hat{P}_{12}(\hat{F}_{1\omega}(1)\hat{F}_0(2)$$
$$+ \hat{F}_0(1)\hat{F}_{1\omega}(2))\ \hat{F}(\mathbf{q})],\ \delta_{+}(x) = {}^1\!/_2\,\delta(x) + i/2\pi x.$$

If we now want to obtain the transport equation it is sufficient to substitute into the right hand side of Eq. (9) the correlation operator $\hat{G}$ given in this way. We restrict ourselves for the moment to the case of a spatially homogeneous distribution. The operator $\hat{F}$ is then diagonal in the p-representation and we have in agreement with Eq. (6)

$$(\mathbf{P}\,|\,\hat{F}\,|\,\mathbf{P}') = N^{-1}\,\delta\,(\mathbf{P}-\mathbf{P}')\,f(\mathbf{p},t).$$

Since the operator $\hat{F}_0$ does not depend on the time, its matrix elements can depend only on the generalized momenta,

$$(\mathbf{P}\,|\,\hat{F}_0\,|\,\mathbf{P}') = N^{-1}\delta\,(\mathbf{P}-\mathbf{P}')\,f_0\,(P).$$

On the other hand, the distribution function can naturally be written in the form

$$f\,(\mathbf{p},t) = f_0\,(p) + f_{1\omega}\,(\mathbf{p})\,e^{i\omega t} + f_{1\omega}\,(\mathbf{p})\,e^{-i\omega t}.$$

Then, in accordance with (11)

$$(\mathbf{P}\,|\,\hat{F}_{1\omega}\,|\,\mathbf{P}') = N^{-1}\delta\,(\mathbf{P}-\mathbf{P}')\,[f_{1\omega}\,(\mathbf{p}) + \tfrac{e}{c}\,\mathbf{A}_0\partial f_0/\partial\mathbf{p}].$$

For the sake of simplicity we shall drop henceforth the index ω. Equation (9) has in the zeroth approximation (with respect to the field) the form

$$\sum_{k=1}^{S}\underset{(k)}{\mathrm{Sp}}[\hat{U}\,(1,\mathbf{q}_k),\ \hat{G}_0\,(1,\mathbf{q}_k)] = 0.$$

Going over from a sum to an integral we have in the p-representation

$$V\,(2\pi\hbar)^{-3}\int\limits_{|\mathbf{q}|\leqslant q_0} d\mathbf{q}\sum_{N_\mathbf{q}=0}^{\infty}(\mathbf{P},N_\mathbf{q}\;|\,[\hat{U}\,(1,\mathbf{q}),\hat{G}_0\,(1,\mathbf{q})]\,|\,\mathbf{P}',N_\mathbf{q}) = 0.$$

The limiting phonon momentum q_0 is connected with the Debye temperature Θ of the material in the well-known way: $h\nu_{q_0} = uq_0 = k\Theta$. After simple, but rather lengthy calculations the equation of the zeroth approximation is of the form

$$I^{(\mathrm{cl})}\,(f_0\,(P)) = 0, \tag{14}$$

where

$$I^{(\mathrm{cl})}\,(f\,(\mathbf{p})) = \frac{\Delta}{(2\pi\hbar)^3}\frac{\pi C^2}{Mu\hbar}\int\limits_{|\mathbf{q}|\leqslant q_0} d\mathbf{q}\;q\,\{\delta\,(a)\,[\overline{N}_q f\,(\mathbf{p}+\mathbf{q})\,(1-f\,(\mathbf{p})) - (\overline{N}_\mathbf{q}+1)\,f\,(\mathbf{p})\,(1-f\,(\mathbf{p}+\mathbf{q}))]$$

$$+\,\delta\,(b)\,[(\overline{N}_\mathbf{q}+1)\,f\,(\mathbf{p}+\mathbf{q})\,(1-f\,(\mathbf{p})) - \overline{N}_\mathbf{q} f\,(\mathbf{p})\,(1-f\,(\mathbf{p}+\mathbf{q}))]\},\quad a = E_p - E_{\mathbf{q}+\mathbf{p}} - h\nu_q,\ \ b = E_p - E_{\mathbf{p}+\mathbf{q}} + h\nu_q$$

is the usual collision integral for electron-phonon interaction. It is well known that this equation is satisfied by the Fermi distribution function

$$f_0\,(p) = [\exp\,(E_p - E_0)\,/\,kT + 1]^{-1}.$$

In the case of a spatially homogeneous distribution we have $[\hat{H}(1),\ \hat{F}(1)] = 0$. From Eq. (9) it follows thus in first approximation that

$$-\,i\omega\hat{F}_1\,(1) = \sum_{k=1}^{S}\underset{(k)}{\mathrm{Sp}}[\hat{U}\,(1,\mathbf{q}_k),\,\hat{G}_1\,(1,\mathbf{q}_k)].$$

Calculations which in this case are even more cumbersome than in the zeroth approximation lead to the following result

$$i\omega\,f_1\,(\mathbf{p}) + i\omega\frac{e}{c}\,\mathbf{A}_0\frac{\partial f_0}{\partial\mathbf{p}} = \hat{I}\,\left(f_1\,(\mathbf{p}) + \frac{e}{c}\,\mathbf{A}_0\,\frac{\partial f_0}{\partial\mathbf{p}}\right) + \frac{e}{c}\,(\mathbf{A}_0\mathbf{p})\,D\,(p). \tag{15}$$

where $\hat{I}$ is the quantum mechanical generalization of the usual collision integral,

$$\hat{I}f_1\,(\mathbf{p}) = \frac{\Delta}{(2\pi\hbar)^3}\,\frac{\pi C^2}{Mu\hbar}\int\limits_{|\mathbf{q}|\leqslant q_0} d\mathbf{q}\cdot q\,\Big\{\frac{1}{2}\,[\delta\,(a+\hbar\omega) + \delta\,(a-\hbar\omega)]\,[-(\overline{N}_\mathbf{q}+1)\,f_1\,(\mathbf{p}) + \overline{N}_q f_1\,(\mathbf{p}+\mathbf{q}) + f_0\,(\mathbf{p})\,f_1\,(\mathbf{p}+\mathbf{q}) +$$

$$+ f_0(\mathbf{p}+\mathbf{q}) f_1(\mathbf{p})] + \frac{1}{2}[\delta(b+\hbar\omega)+\delta(b-\hbar\omega)]\,[(\overline{N}_q+1) f_1(\mathbf{p}+\mathbf{q}) - \overline{N}_q f_1(\mathbf{p}) - f_1(\mathbf{p}+\mathbf{q}) f_0(p) - f_1(\mathbf{p}) f_0(\mathbf{p}+\mathbf{q})]\Big\};$$

$$D(p) = + \frac{\Delta}{(2\pi\hbar)^3}\,\frac{\pi C^2}{2mMu\hbar p_0^2}\,\frac{1}{\hbar\omega}\,(e^{\hbar\omega/kT}-1) f_0(p)\, e^{(E_p-E_0)/kT} \int\limits_{|\mathbf{q}|\leqslant q_0} d\mathbf{q}\, q\,(\mathbf{pq})\,\overline{N}_q f_0(\mathbf{p}+\mathbf{q})\,[\delta(a+\hbar\omega)$$

$$+ e^{-h\omega/kT}\delta(a-\hbar\omega) + e^{h\nu_q/kT}\delta(b+\hbar\omega) + e^{h\nu_q-\hbar\omega/kT}\delta(b-\hbar\omega)].$$

For $\hbar\omega \to 0$, $\hat{I}(f_1)$ goes over into $I^{(cl)}(f_1)$. In order to understand the meaning of the rest of the terms on the right hand side of Eq. (15) we notice that the last of them can be written in the form

$$\frac{e}{c}(\mathbf{A}_0\mathbf{p})\, D(\mathbf{p}) = \frac{\Delta}{(2\pi\hbar)^3}\,\frac{\pi C^2}{Mu\hbar}\,\frac{e}{mc} \int\limits_{|\mathbf{q}\leqslant q_0} d\mathbf{q}\cdot q\,(\mathbf{A}_0\mathbf{q}) \Big\{\frac{1}{2\hbar\omega}[\delta(a+\hbar\omega)-\delta(a-\hbar\omega)]\,[-(\overline{N}_{\mathbf{q}}+1) f_0(p)(1-f_0(\mathbf{p}+\mathbf{q}))$$

$$+ \overline{N}_q f_0(\mathbf{p}+\mathbf{q})(1-f_0(p))] + \frac{1}{2\hbar\omega}[\delta(b+\hbar\omega)-\delta(b-\hbar\omega)]\,[-N_q f_0(p)(1-f_0(\mathbf{p}+\mathbf{q})) + (\overline{N}_q+f) f_0(\mathbf{p}+\mathbf{q})(1-f_0(p))]\Big\}.$$

Let us imagine further that in the equation $I^{(cl)}(f_0(p)) \equiv 0$ we have substituted $p = \mathbf{P} + e\mathbf{A}_0/c$ and have carried out an expansion in terms of $\mathbf{A}_0$ accurate up to terms of the first order of magnitude. Then $I^{(cl)}(f_0(p))$ can be written as the sum of three expressions; the first of these is the same as expression (14) and is equal to zero, the second arises from expanding expressions of the kind $f_0(\mathbf{P} + e\mathbf{A}/c)$ and the third one, finally, arises from the expansion of the δ-function. It is clear that the terms

$$\hat{I}\,\frac{e}{c}\,\mathbf{A}_0\frac{\partial f_0}{\partial \mathbf{p}} \text{ and } \frac{e}{c}(\mathbf{A}_0.\mathbf{p}) D(\mathbf{p})$$

are the quantum mechanical generalizations of the last two expressions and cancel one another as $\hbar\omega \to 0$.

In the case of a spatially inhomogeneous distribution the method which we have just applied to obtain the transport equation will, generally speaking, not be suitable, since now in the transformation (5) the vector potential depends essentially on the coordinates, and, in particular, from Eq. (6) there does not follow a simple connection between $F_{\mathbf{p},\mathbf{p}'}$ and the function $f(\mathbf{p}, \mathbf{r}, t)$. However, it is necessary to take into account the fact that the distance over which the field or the electron distribution changes materially will always be very large compared to the de Broglie wavelength of an electron near the Fermi surface. In other words, if we write, for instance, $f_1(\mathbf{p}, \mathbf{r}, t)$ as a Fourier integral of the coordinates,

$$f_1(\mathbf{p}, \mathbf{r}, t) = \int f_{1\mathbf{k}}(\mathbf{p}, t)\, e^{i\mathbf{kr}/\hbar}\, d\mathbf{k},$$

$f_{1\mathbf{k}}$ will be noticeably different from zero only, if

$$|\mathbf{k}| \ll p_0. \tag{16}$$

For this reason all terms of Eq. (15) retain their form with the proviso that now $\mathbf{A}_0$ and f_1 will be functions of the coordinate $\mathbf{r}$. We only have to take into account the term $[\hat{H}(1), \hat{F}(1)]$ of Eq. (9) which previously vanished.

It can easily be shown that the left hand side of the transport equation can be reduced to the expression

$$\frac{\mathbf{p}}{m}\,\frac{\partial f_1}{\partial \mathbf{r}} + i\,\frac{e}{mc}\,\frac{1}{(2\pi\hbar)^3}\int d\tau\, d\mathbf{p}' e^{-i\tau(\mathbf{p}-\mathbf{p}')/\hbar} f_0(p') \left[\left(\tau, \frac{\partial}{\partial \mathbf{r}}\right)(\mathbf{p}.\,\mathbf{A}_0(\mathbf{r})) - \left(\mathbf{p}'.\,\mathbf{A}_0\left(\mathbf{r}+\frac{\tau}{2}\right) - \mathbf{A}_0\left(\mathbf{r}-\frac{\tau}{2}\right)\right)\right],$$

in which the integral terms can be neglected by virtue of condition (16).

Introducing the electric field strength $\mathbf{E}(\mathbf{r}) = (i\omega/c)\,\mathbf{A}_0(\mathbf{r})$ and the relaxation time operator $\hat{\tau} = -\hat{I}^{-1}$ we write the transport equation finally in the form

$$i\omega f_1 + \frac{\mathbf{p}}{m}\,\frac{\partial f_1}{\partial \mathbf{r}} + \hat{\tau}^{-1} f_1 = e\mathbf{E}\left[\frac{\partial f_0}{\partial \mathbf{p}} + \frac{1}{i\omega\hat{\tau}}\,\frac{\partial f_0}{\partial \mathbf{p}} - \frac{1}{i\omega}\,\mathbf{p}D(p)\right]. \tag{17}$$

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TRANSPORT PHENOMENA IN A COMPLETELY IONIZED TWO-TEMPERATURE PLASMA*

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A system of transport equations has been obtained for a plasma consisting of electrons and one kind of positive ions placed in an electric and magnetic field. The system includes the continuity equations, equations of motion, and the equation of heat transport for electrons and ions. The electron and ion temperatures are considered to be different. The case of arbitrary ratio of the particle collision frequency to the Larmor frequency is considered. The derivation of the transport equations from the kinetic equations is similar to that of Chapman and Cowling.

1. THE TRANSPORT EQUATIONS

THE state of a completely ionized plasma can be characterized by the electron and ion distribution functions $f_\alpha(t, \mathbf{r}, \mathbf{v})$. In the presence of electric and magnetic fields **E** and **H** these distribution functions satisfy the following system of kinetic equations (see, for example, Chapman and Cowling[1])

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\nabla f_\alpha + \frac{e_\alpha}{m_\alpha}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}]\right)\nabla_v f_\alpha + \sum_\beta S_{\alpha\beta}(f_\alpha f_\beta) = 0, \tag{1.1}$$

where α denotes the type of particle (1 — electrons, 2 — ions).

Following Landau,[2] we take the collision integrals $S_{\alpha\beta}(f_\alpha f_\beta)$, which give the change in the distribution of particles of type α as a result of their collisions with particles of type β, to be of the form:

$$S_{\alpha\beta}(f_\alpha f_\beta) = \frac{2\pi\lambda e_\alpha^2 e_\beta^2}{m_\alpha}\frac{\partial}{\partial v_i}\int\left\{\frac{f_\alpha}{m_\beta}\frac{\partial f'_\beta}{\partial v'_k} - \frac{f'_\beta}{m_\alpha}\frac{\partial f_\alpha}{\partial v_k}\right\}U_{ik}d\mathbf{v}', \tag{1.2}$$

where

$$U_{ik} = (u^2\delta_{ik} - u_i u_k)/u^3, \quad u_i = v_i - v'_i.$$

The "Coulomb logarithm" λ appearing in (1.2) is equal to the logarithm of the ratio of the maximum and minimum impact parameters $\lambda = \ln(p_{max}/p_{min})$. For the smaller impact parameter one should substitute into the above $p_{\min} \approx e^2/mv^2 \approx e^2/T$. The maximum impact parameter is determined by the fact that the Coulomb field of particles in a plasma is screened at distances of the order of a Debye length $p_{\max} \approx \delta_D = (T/4\pi e^2 n)^{1/2}$. At high speeds when $e^2/hv < 1$ (or $v/c > 1/137$) a smaller value should be taken for the maximum impact parameter, i.e., one for which the scattering angle becomes of the same order as its quantum-theoretical indeterminacy $p_{\max} = \delta_D\, e^2/hv$. The influence of the magnetic field on the collision event itself is not taken into account in (1.2), which is permissible only for not too strong fields when the radius of curvature of the particle trajectory is large compared to a Debye length.

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*Work performed in 1952.

The method of obtaining the transport equations, starting from the kinetic equations, is given in detail in Chapman and Cowling's monograph,[1] where expressions are also obtained for the heat flow and for the stress tensor for a single-component ionized gas in a magnetic field, and also for the electrical conductivity of a plasma in a magnetic field. The electrical conductivity of plasma (in particular, of a multicomponent one) in a magnetic field is investigated in greater detail by Cowling[3] by the same method. The method described by Chpman and Cowling[1] assumes that the temperatures of all the components of the plasma are the same. A calculation of the electrical conductivity and of the electronic heat conductivity of the plasma was carried out by Landshoff[4] without assuming that the temperatures of the ions and the electrons are equal for not very strong magnetic fields (the ratio of the Larmor frequency to the collision frequency lying between zero and six). The transport equations for a plasma in a magnetic field of arbitrary intensity were obtained by Fradkin[5] for the case of equal electron and ion temperatures.

In this paper the method given by Chapman and Cowling[1] is somewhat modified to obtain a separate system of transport equations for each plasma component. The microscopic parameters (density, average velocity, and temperature) for the ions and the electrons are given by the following expressions:

$$n_\alpha(t, \mathbf{r}) = \int f_\alpha(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v}, \quad \mathbf{v}_\alpha^0(t, \mathbf{r}) = \frac{1}{n_\alpha}\int \mathbf{v} f_\alpha(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v}, \quad \frac{3}{2} T_\alpha(t, \mathbf{r}) = \frac{1}{n_\alpha}\int \frac{m_\alpha}{2}(\mathbf{v} - \mathbf{v}_\alpha^0)^2 f_\alpha(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v}. \tag{1.3}$$

The temperatures of the ions and of the electrons are, generally speaking, taken to be different. This turns out to be possible because of the large difference in the masses of the ions and the electrons. Thus, plasma is regarded as consisting of two mutually interpenetrating fluids — the electron and the ion fluids — similarly to the way in which this has already been done in the less refined investigation of Schlüter.[6]

For subsequent developments it is convenient to introduce into (1.1) the change of variables $\mathbf{v}_\alpha = \mathbf{v} - \mathbf{v}_\alpha^0(t, \mathbf{r}_\alpha)$. We then obtain for $f_\alpha(t, \mathbf{r}, \mathbf{v}_\alpha)$ the equation

$$\frac{d_\alpha f_\alpha}{dt} + \mathbf{v}\nabla f_\alpha + \left(\frac{e_\alpha}{m_\alpha}\mathbf{E} + \frac{e_\alpha}{m_\alpha c}[\mathbf{v}_\alpha^0 \times \mathbf{H}] - \frac{d_\alpha \mathbf{v}_\alpha^0}{dt}\right)\nabla_v f_\alpha - \frac{\partial v_{\alpha l}^0}{\partial x_k} v_k \frac{\partial f_\alpha}{\partial v_i} + \frac{e_\alpha}{m_\alpha c}[\mathbf{v}\times\mathbf{H}]\nabla_v f_\alpha + \sum_\beta S_{\alpha\beta}(f_\alpha f_\beta) = 0. \tag{1.4}$$

Here $d_\alpha/dt = \partial/\partial t + \mathbf{v}_\alpha^0 \nabla$, and the index α has been omitted from $\mathbf{v}_\alpha$ for the safe of brevity.

Multiplication of (1.4) in turn by 1, $m_\alpha \mathbf{v}_\alpha m_\alpha \mathbf{v}_\alpha^2/2$ and integration over the velocities leads to the following system of equations for the macroscopic parameters of the gas α

$$d_\alpha n_\alpha / dt + n_\alpha \operatorname{div} \mathbf{v}_\alpha^0 = 0, \quad m_x n_\alpha \frac{d_\alpha \mathbf{v}_\alpha^0}{dt} = -\operatorname{grad} p_\alpha - \operatorname{div} \pi_\alpha + e_\alpha n_\alpha\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_\alpha^0 \times \mathbf{H}]\right) + \mathbf{R}_\alpha, \quad \frac{3}{2} n_\alpha \frac{d_\alpha T_\alpha}{dt} + p_\alpha \operatorname{div} \mathbf{v}_\alpha^0$$
$$+ \pi_{\alpha ik}\frac{\partial v_{\alpha l}^0}{\partial x_k} + \operatorname{div} \mathbf{q}_\alpha = Q_\alpha. \tag{1.5}$$

The first of these equations is the equation of continuity, the second is the equation of momentum transfer, and the third is the equation of heat transfer. The following average quantities have been introduced:

$$p_\alpha = \int (m_\alpha v_\alpha^2/3) f_\alpha d\mathbf{v} = n_\alpha T_\alpha, \qquad \pi_{\alpha ik} = \int m_\alpha \left(v_{\alpha i} v_{\alpha k} - \frac{1}{3}\delta_{ik} v_\alpha^2\right) f_\alpha d\mathbf{v}, \qquad \mathbf{q}_\alpha = \int (m_\alpha v_\alpha^2/2)\, \mathbf{v}_\alpha f_\alpha d\mathbf{v},$$

$$\mathbf{R}_\alpha = -\int m_\alpha \mathbf{v}_\alpha S_{\alpha\beta}(f_\alpha f_\beta)\, d\mathbf{v}, \quad Q_\alpha = -\int (m_\alpha v_\alpha^2/2)\, S_{\alpha\beta}(f_\alpha f_\beta)\, d\mathbf{v}.$$

Here p_α is the partial pressure of particles α. We shall denote the symmetric tensor π_{ik} as the viscosity tensor. The vector $\mathbf{q}_\alpha$ is the heat flux for particles of type α; R_α and Q_α are respectively the average rate of change of the momentum of particles α and the rate of liberation of heat in the gas formed by particles α due to their collisions with particles β.

Equations (1.5) may be used for finding the macroscopic parameters after first finding the local distribution functions and expressing the quantities (1.6) in terms of the macroscopic parameters and their derivatives. The following sections are devoted to this. The expressions obtained as a result of the approximate solution of the kinetic equations are given in (2.3), (2.5), (2.6), (3.16), (3.18), (3.19), (3.20), (4.13), (4.14′), and (4.16) while the necessary coefficients are given in a table (see p. 364).

As in the case of the usual equations of gas dynamics, the condition for the applicability of these expressions is that all these quantities should vary but little over distances on the order of a mean free path and during times on the order of the mean collision time between particles. This holds for not too high values of the ratio of the Larmor frequency to the collision frequency $\omega\tau \lesssim 1$. But in the case of strong magnetic fields when $\omega\tau \gg 1$ the problem becomes more complicated. For the applicability of the results obtained it is necessary that during the time between two collisions the particle be displaced through a distance so small that all quantities undergo only a small change over it. The motion of the particle transversely to the magnetic field is limited by its Larmor radius ρ which is less than the mean free path ℓ by a factor $\omega\tau$. Therefore in a number of cases the conditions of applicability for processes close to stationary ones are relaxed, and we require only that $L_\perp \gg \rho$, $L_\parallel \gg \ell$ ($L_\perp$, $L_\parallel$ are characteristic distances in directions perpendicular and parallel to the magnetic field over which the various quantities vary significantly). However, this is true only in the presence of a sufficient degree of symmetry. In an inhomogeneous magnetic field (and also in the presence of a transverse electric field) in addition to the Larmor rotation in a circle the particles also undergo a drifting motion across the magnetic field.[7] This drift occurs with a velocity of the order of $v\rho/L_\perp$ (v is the thermal velocity) and leads to a displacement of the particle during the time between collisions by a distance of the order of $\ell\rho/L_\perp$. Therefore the condition $L_\perp \gg \rho$ is sufficient only in the case when the system possesses such symmetry that the particle drifts are not directed along gradients of density, temperature, etc. If such a special symmetry does not exist then the stricter condition is obtained $L_\perp \gg \ell\rho/L_\perp$, or $L_\perp \gg \sqrt{\ell\rho}$.

2. METHOD OF APPROXIMATE SOLUTION OF THE KINETIC EQUATIONS

The ion and electron distribution functions satisfy a system of kinetic equations. By taking into account the smallness of the ratio of the electron mass to the ion mass we can simplify the problem and solve these equations separately. This is connected with the fact that the electron velocities are much higher than the ion velocities so that the relative velocity of the electron and the ion nearly coincides with the electron velocity. Therefore, the cross term S_{12} is to a high degree of accuracy independent of the detailed shape of the ion distribution function, but is determined by specifying the average values of n_2, $\mathbf{v}_2^0$, and T_2 for the ions.

The tensor U_{ik} in the electron-ion collision integral may be expanded in powers of the ion velocity and integrated over the ion velocities. Neglecting the viscosity tensor for the ions we obtain:

$$S_{12} = -\frac{2\pi\lambda e_1^2 e_2^2 n_2}{m_1^2} \frac{\partial}{\partial v_i} \left\{ V_{ik} \frac{\partial f_1}{\partial v_k} + \frac{m_1}{m_2} \left(\frac{2v_i}{v^3} f_1 + \frac{T_2}{m_1} \frac{3v_i v_k - v^2\delta_{ik}}{v^5} \frac{\partial f_1}{\partial v_k} \right) \right\}, \quad V_{ik} = (v^2\delta_{ik} - v_i v_k)/v^3. \tag{2.1}$$

The electron velocity is here referred to the mean ion velocity. We shall denote the first (principal) term in (2.1) by S'_{12}. It reduces to zero for any spherically symmetric electron distribution function and does not give any change in the electron energy. Later we shall need an expression for the friction force acting on the electrons due to the presence of the ions in the case when the electrons have a Maxwellian distribution (we denote it by $f_1^{(0)}$) displaced with respect to the ion distribution by the amount $\mathbf{u} = \mathbf{v}_1^0 - \mathbf{v}_2^0$. Assuming this shift to be small compared to the electron thermal velocities and neglecting terms $\sim m_1/m_2$ we shall easily obtain with the aid of (2.1)

$$\mathbf{R}_1^{(0)} = -\int m_1 \mathbf{v} S'_{12} (f_1^{(0)})\, d\mathbf{v} = -(m_1 n_1 / \tau_1)\, \mathbf{u}. \tag{2.2}$$

Here we have introduced the "time between collisions" between electrons and ions

$$\tau_1 = 3\sqrt{m_1} T_1^{3/2} / 4\sqrt{2\pi}\, \lambda e_1^2 e_2^2 n_2. \tag{2.3}$$

The ion-electron collision integral can also be simplified by expanding U_{ik} in terms of the ratio of the ion speed to the electron speed. Assuming that the electron distribution function differs but little from a Maxwellian one, with the difference between the mean velocities of the electrons and the ions being small compared to the electron velocities, we obtain:

$$S_{21} = -\frac{m_1 n_1}{m_2 n_2}\frac{1}{\tau_1}\frac{\partial}{\partial v_i}\left(v_i f_2 + \frac{T_1 \partial f_2}{m_2 \partial v_i}\right) + \frac{1}{m_2 n_2}\mathbf{R}_2 \nabla_v f_2. \tag{2.4}$$

The ion velocity is here referred to their average velocity. This collision term, as should have been expected, has the same structure as in the case of brownian particles in a moving medium.

The transfer of heat from the electrons to the ions by means of collisions can be easily calculated with the aid of (2.4). By neglecting the deviation of the ion distribution function from a Maxwellian one we obtain:

$$Q_2 = \frac{3m_1 n_1}{m_2 \tau_1}(T_1 - T_2). \tag{2.5}$$

This expression has been obtained by Landau.^{2} The liberation of heat in the electron gas can be most simply computed by making use of the conservation of energy in collisions. Simple calculations yield:

$$Q_1 = -\mathbf{R}_1 \mathbf{u} - Q_2. \tag{2.6}$$

By evaluating the corresponding integrals or by considering directly collisions between particles it may be easily seen that if the energies of the light and of the heavy particles are of the same order of magnitude then the times for the exchange of energy for the light particles (electrons) among themselves, and also for the heavy particles (ions) are much shorter than that between the electrons and the ions. Let the time for energy exchange between electrons be τ_1, the time for energy exchange between ions be τ_2, and the time for energy exchange between electrons and ions be τ_3, then:

$$\tau_1 : \tau_2 : \tau_3 = 1 : \sqrt{m_2/m_1} : m_2/m_1. \tag{2.7}$$

It is this relation which allows us to treat the problem with ions and electrons of different temperatures, since equilibrium is established within each gas more rapidly than that between them. The transfer of momentum from the electrons to the ions, in contrast to the transfer of energy, is not small and occurs during times of the same order of magnitude as the exchange of momentum between electrons. The transfer of the ion momentum to the electrons occurs during the same time as the transfer of energy, and is therefore small compared to the exchange of momentum between the ions.

Thus for the ions the cross collision integral is small compared with the self integral $S_{21} \ll S_{22}$. But for the electrons although the cross integral is not small it may be represented as the sum of two integrals $S_{12} = S'_{12} + S''_{12}$ such that the first of them gives only the transfer of momentum but does not lead to a change in the electron energy, while the second one is small (see 2.1). The kinetic equation for the electrons may be written in the following form:

$$S_{11}(f_1 f_1) + S'_{12}(f_1 f'_2) + [\mathbf{v}\times\boldsymbol{\omega}_1]\nabla_v f_1 = -\left\{\frac{d_1 f_1}{dt} + \mathbf{v}\nabla f_1 + \left(\frac{e_1}{m_1}\mathbf{E} + \frac{e_1}{m_1 c}[\mathbf{v}_1^0 \times \mathbf{H}] - \frac{d_1 \mathbf{v}_1^0}{dt}\right)\nabla_v f_1 - \frac{\partial v_{1i}^0}{\partial x_k} v_k \frac{\partial f_1}{\partial v_i} + S'_{12}(f_1, f_2 - f'_2) + S''_{12}(f_1 f_2)\right\}. \tag{2.8}$$

The terms on the right hand side of this equation are small when the gradients are small, the time variations are slow and the shift of the mean velocities of the electrons and the ions is small. The magnitude of the vector $\boldsymbol{\omega}_1 = (e_1/mc)\mathbf{H}$ is equal to the Larmor frequency of the electrons (we note that $e_1 < 0$, $\omega_1 < 0$). The term $S'_{12}(f_1 f'_2)$ has been added to and substracted from equation (2.8) in which f'_2 is the ion distribution function "shifted" in such a way that the mean velocity of the ions coincides with that of the electrons. The term $S'_{12}(f_1, f_2 - f'_2)$ which appears in the course of this on the right hand side of the equation is small compared to $S'_{12}(f_1, f'_2)$ provided that the relative macroscopic velocity between the electrons and the ions is small compared to the thermal velocities of the electrons.

The zero-order approximation satisfies the equation without the right hand side. The solution of this equation is arbitrary Maxwellian distribution:

$$f_1^{(0)} = n_1 (2\pi T_1/m_1)^{-3/2} \exp(-m_1 v^2/2T_1). \tag{2.9}$$

The magnetic field evidently has no influence on the Maxwellian distribution.

We shall assume that the parameters of this distribution correspond to the density and the temperature at the given point; the velocity in (2.9) is referred to its mean value $\mathbf{v}_1^0$. In seeking now a correction to this Maxwellian distribution we shall require that it should not change the values of these macroscopic parameters.

Let the electron distribution function have the form $f_1 = f_1^{(0)}(1 + \Phi)$ where Φ is a small correction. By substituting this expression into (2.8) and discarding terms of the second order of smallness we shall obtain the equation which determines the correction. We shall replace the time derivatives of n, $\mathbf{v}^0$, and T appearing on the right hand side by their zero-order approximations. Without its right hand side the equation has the solutions $\Phi = 1, v^2$, and therefore in order for the equation to be soluble its right hand side must be orthogonal to these solutions. By multiplying the equation for the correction by 1, **v**, $mv^2/2$ and by integrating over the velocities, by taking into account the condition $\int \mathbf{v} f_1^0 \Phi \, d\mathbf{v} = 0$ we shall obtain the expression for the zero order approximations for the time derivatives n, $\mathbf{v}^0$, T which must be substituted into the right hand side. The whole procedure is analogous to the one developed by Chapman and Cowling.[1]

As a result we obtain for the first-order correction:

$$I_1(\Phi) + I'_{12}(\Phi) + f_1^{(0)}[\mathbf{v}\times\boldsymbol{\omega}_1]\nabla_v\Phi = -f_1^{(0)}\left\{\left(\frac{m_1 v^2}{2T_1} - \frac{5}{2}\right)\mathbf{v}\nabla\ln T_1 + \frac{m_1}{2T_1}\left(v_i v_k - \frac{v^2}{3}\delta_{ik}\right)w_{1ik} + \frac{\mathbf{R}_1^{(1)}\mathbf{v}}{n_1 T_1} + \left(\frac{3\sqrt{\pi}}{\sqrt{2}}\frac{(T_1/m_1)^{3/2}}{v^3} - 1\right)\frac{m_1}{T_1}\frac{\mathbf{u}\mathbf{v}}{\tau_1}\right\}. \tag{2.10}$$

Here we have introduced the notation:

$$I_1(\Phi) = S_{11}(f_1^{(0)}, f_1^{(0)}\Phi) + S_{11}(f_1^{(0)}\Phi, f_1^{(0)}),\; I'_{12}(\Phi) = S'_{12}(f_1^{(0)}\Phi,\; f'_2); \tag{2.11}$$

$$\mathbf{R}_1^{(1)} = -\int m_1 \mathbf{v} S'_{12}(f_1^{(0)}\Phi,\; f'_2)\, d\mathbf{v}. \tag{2.12}$$

In the right hand side of (2.10) the terms $\sim m_1/m_2$ have been omitted. The integral $S'_{12}(f_1^{(0)}, f_2 - f'_2)$ has been expanded in powers of $u/\sqrt{T_1/m_1}$ (this quantity is assumed to be small) and only the first term of the expansion has been retained. In the second term in curly brackets symmetrization has been performed and a symmetric tensor with zero trace has been introduced

$$w_{ik} = \partial v_i^0/\partial x_k + \partial v_k^0/\partial x_i - 2/3\,\delta_{ik}\,\mathrm{div}\,\mathbf{v}^0. \tag{2.13}$$

The kinetic equation for the ions is transformed in a similar way with, however, the one difference that the whole cross collision term [it is taken to be of the form (2.4)] is regarded as small and is transferred to the right-hand side. The zero-order approximation to the ion distribution function which satisfies the equation without its right-hand side is the Maxwellian distribution $f_2^{(0)}$. The ion distribution function is represented in the form $f_2 = f_2^{(0)}(1 + \Phi)$ and the following equation is obtained for the small correction Φ:

$$I_2(\Phi) + f_2^{(0)}[\mathbf{v}\times\boldsymbol{\omega}_2]\nabla_v\Phi = -f_2^{(0)}\left\{\left(\frac{m_2 v^2}{2T_2} - \frac{5}{2}\right)\mathbf{v}\nabla\ln T_2 + \frac{m_2}{2T_2}\left(v_i v_k - \frac{v^2}{3}\delta_{ik}\right)w_{2ik}\right\}. \tag{2.14}$$

The terms on the right-hand side associated with collisions with electrons are mutually cancelled. As a result Eq. (2.14) has the same form as for a simple gas (not a mixture). Thus the form of the ion distribution function in the approximation under consideration is determined only by collision of ions with ions. On the contrary the form of the electron distribution function is determined both by self (electron-electron) collisions, and by cross (electron-ion) collisions.

3. HEAT FLOW AND TRANSFER OF MOMENTUM

The solution of Eq. (2.10) may be sought in the form

$$\Phi(\mathbf{v}) = \Phi_i(v)\, v_i + \Phi_{ik}(v)\left(v_i v_k - \frac{v^2}{3}\delta_{ik}\right). \tag{3.1}$$

The first term may be represented in the form: $\Phi_i v_i = \Phi_u + \Phi_T$, where

$$\Phi_u = A_u \mathbf{v}\mathbf{u}_\parallel + A'_u \mathbf{v}\mathbf{u}_\perp + A''_u \mathbf{v}[\boldsymbol{\omega}_1\times\mathbf{u}]; \tag{3.2}$$

$$\Phi_T = A_T \mathbf{v}\nabla_\parallel \ln T_1 + A'_T \mathbf{v}\nabla_\perp \ln T_1 + A''_T \mathbf{v}[\boldsymbol{\omega}_1\times\nabla\ln T_1]. \tag{3.3}$$

The coefficients A are functions of the absolute value of the velocity. The subscript ∥ denotes the component of the corresponding vector along the magnetic field while ⊥ denotes the perpendicular component. It is evidently sufficient to find A′ and A″; the expression for A is obtained from A′ by setting $\omega_1 = 0$.

Knowing Φ_i one can find the heat flow and the momentum transfer due to collisions (the friction force and the thermal force).

Hereinafter it will be convenient to utilize the expansion of the functions A into a series of Sonine polynomials of order 3/2. The Sonine polynomials have the following orthogonality property:

$$\int_0^\infty e^{-x} L_p^{(3/2)}(x) L_q^{(3/2)}(x) x^{3/2} dx = \frac{\Gamma(p + 5/2)}{p!} \delta_{pq}. \tag{3.4}$$

In place of the velocity **v** we introduce the dimensionless variable

$$\mathbf{s} = (m_1 / 2T_1)^{1/2} \mathbf{v}. \tag{3.5}$$

We write the expansion of A'_u, A''_u in terms of Sonine polynomials in the form:

$$A'_u = -\frac{m_1}{T_1} \sum_{k=1}^{\infty} a'_k L_k^{(3/2)}(s^2), \quad A''_u = -\frac{m_1}{T_1} \sum_{k=1}^{\infty} a''_k L_k^{(3/2)}(s^2). \tag{3.6}$$

The orthogonality relation (3.4) leads to the fact that the heat flux is determined by the first expansion coefficient

$$\mathbf{q}_u = {}^5/_2\, n_1 T_1 (a'_1 \mathbf{u}_\perp + a''_1 [\boldsymbol{\omega}_1 \times \mathbf{u}]). \tag{3.7}$$

The transfer of momentum by collisions (friction force) will be expressed in the following way:

$$\mathbf{R}_u^{(1)} = {}^5/_2 \frac{m_1 n_1}{\tau_1} \sum_{k=1}^{\infty} \alpha'_{0k} (a'_k \mathbf{u}_\perp + a''_k [\boldsymbol{\omega}_1 \times \mathbf{u}]). \tag{3.8}$$

Here the following notation has been introduced for the matrix elements:

$$\alpha'_{0k} = \frac{4\tau_1}{15 n_1} \int s_i I'_{12} (L_k^{(3/2)}(s^2) s_i) \, d\mathbf{v}, \tag{3.9}$$

where τ_1 is the time between collisions [see (2.3)]. The matrix elements (3.9) are calculated in the Appendix.

In a similar way the heat flux and the transfer of momentum due to a temperature gradient may be expressed in terms of the expansion coefficients. On writing A'_T, A''_T in the form

$$A'_T = \tau_1 \sum_{k=1}^{\infty} a'_k L_k^{(3/2)}(s^2), \quad A''_T = \tau_1 \sum_{k=1}^{\infty} a''_k L_k^{(3/2)}(s^2), \tag{3.10}$$

we obtain:

$$\mathbf{q}_T = -\frac{5}{2} \frac{n_1 T_1 \tau_1}{m_1} (a'_1 \nabla_\perp T_1 + a''_1 [\boldsymbol{\omega}_1 \times \nabla T_1]); \tag{3.11}$$

$$\mathbf{R}_T = -\frac{5}{2} n_1 \sum_{k=1}^{\infty} \alpha'_{0k} (a'_k \nabla_\perp T_1 + a''_k [\boldsymbol{\omega}_1 \times \nabla T_1]). \tag{3.12}$$

Of course the a'_k, a''_k are different from those in (3.7), (3.8), and the corresponding subscripts have been omitted merely to reduce the awkwardness of the notation.

The equation which determines that part of the correction which is due to the transverse component of the temperature gradien has the form:

$$I_1(\Phi) + I'_{12}(\Phi) + f_1^{(0)} [\mathbf{v} \times \omega_1] \nabla_v \Phi = -f_1^{(0)} \left\{ \left(s^2 - \frac{5}{2} \right) \mathbf{v} \nabla_\perp \ln T_1 + \frac{\mathbf{R}_T^{(1)} \mathbf{v}}{n_1 T_1} \right\}. \tag{3.13}$$

We represent the thermal force in the form:

$$\mathbf{R}_T^{(1)} = n_1 T_1 (K' \nabla_\perp \ln T_1 + K'' [\omega_1 \times \nabla \ln T_1]).$$

We introduce the complex quantities $A = A' + i\omega_1 A''$, $K = K' + i\omega_1 K''$. For the function A we obtain

$$I_1(As) + I'_{12}(As) + i\omega_1 f_1^{(0)} As = -f_1^{(0)}\left\{\left(s^2 - \frac{5}{2}\right) + K\right\} s. \tag{3.14}$$

Making use of the orthogonality of Sonine polynomials we obtain an infinite system of equations for the expansion coefficients $a_k = a'_k + i\omega_1 a''_k$

$$\sum_{l=1}^{\infty} (\alpha_{kl} + \alpha'_{kl})\, a_l + i\omega_1\tau_1 \frac{\Gamma(k + {}^5/_2)}{k!\,\Gamma({}^7/_2)}\, a_k = \delta_{1k}. \tag{3.15}$$

The matrix elements α_{ik}, α'_{ik} are evaluated in the Appendix.

For an approximate evaluation of the coefficients a_k one can break off the series (3.10) after the first two terms and also break off in a corresponding manner the infinite system of equations (3.15). As a result one obtains a system of two equations which yield the coefficients a'_1, a'_2, a''_1, a''_2 while (3.11) and (3.12) yield the heat flux and the thermal force. For $|\omega_1|\tau_1 \gg 1$ one can in this way correctly obtain the first two terms in the expansions in powers of $(\omega_1\tau_1)^{-1}$ of the coefficients of interest to us. For $|\omega_1|\tau_1 \lessgtr 1$ there is no such formal parameter which characterizes the accuracy of the calculations, but in spite of this one can hope that the coefficients obtained in this way will give a sufficiently good approximation.

The coefficients a_k depend on two dimensinless parameters: the "magnetization coefficient" $\omega_1\tau_1$ and the effective charge of the ions $Z = n_2 e_2^2/n e_1^2 \approx e_2/|e_1|$ (the ratio of the matrix elements α_{ik} and α'_{ik} depends on the latter).

Expressions for the thermal force and for the heat flow due to the temperature gradient have the form:

$$\begin{aligned} \mathbf{R}_{1T} &= -n_1\left\{\gamma_{uT}\nabla_{\parallel} T_1 + \frac{1}{\Delta}(\beta'_{uT}x + \gamma'_{uT})\nabla_{\perp} T_1 - \frac{1}{\Delta}(\beta''_{uT}x + \gamma''_{uT})\,\tau_1\,[\boldsymbol{\omega}_1 \times \nabla T_1]\right\}; \\ \mathbf{q}_{1T} &= -\frac{n_1 T_1 \tau_1}{m_1}\left\{\gamma_{TT}\nabla_{\parallel} T_1 + \frac{1}{\Delta}(\beta'_{TT}x + \gamma'_{TT})\nabla_{\perp} T_1 - \frac{1}{\Delta}(\beta''_{TT}x + \gamma''_{TT})\,\tau_1\,[\boldsymbol{\omega}_1 \times \nabla T_1]\right\}, \end{aligned} \tag{3.16}$$

where

$$\Delta = x^2 + \delta_1 x + \delta_0, \qquad x = \omega_1^2\tau_1^2. \tag{3.16'}$$

The values of the coefficients for Z = 1, 2, 3, 4 are given in the following table:

	$Z=1$	$Z=2$	$Z=3$	$Z=4$
$\gamma_{TT} = \gamma'_{TT}/\delta_0$	3.1616	4.8901	6.0641	6,9200
$\gamma_{Tu} = \gamma'_{Tu}/\delta = \gamma_{uT}$	0.7110	0.9052	1.0158	1,0904
$\gamma_{uu} = 1 - \gamma'_{uu}/\delta_0$	0.5129	0.4408	0.3965	0,3752
δ_0	3.7703	1.0465	0.5814	0,4106
δ_1	14.79	10.80	9.618	9,055
β'_{TT}	4.664	3.957	3.721	3,604
γ'_{TT}	11.92	5.118	3.525	2,841
β''_{TT}	5/2	5/2	5/2	5/2
γ''_{TT}	21.67	15.37	13.53	12.65
$\beta'_{Tu} = \beta'_{uT}$	5.101	4.450	4.233	4.124
$\gamma'_{Tu} = \gamma'_{uT}$	2.681	0.9473	0.5905	0.4478
$\beta''_{Tu} = \beta''_{uT}$	3/2	3/2	3/2	3/2
$\gamma''_{Tu} = \gamma''_{uT}$	3.053	1.784	1.442	1.285
β'_{uu}	6.416	5.523	5.226	5.077
γ'_{uu}	1.837	0.5956	0.3515	0,2566
β''_{uu}	1.704	1.704	1.704	1.704
γ''_{uu}	0.7796	0.3439	0.2400	0.1957

The correction to the electron distribution function, necessitated by the exixtence of a relative velocity, is found in a completely analogous manner. The infinite system of algebraic equations for the complex coefficients $a_k = a'_k + i\omega_1 a''_k$ has the form:

$$\sum_{l=1}^{\infty} (\alpha_{kl} + \alpha'_{kl})\, a_l + \frac{\Gamma(k + {}^5/_2)}{k!\,\Gamma({}^7/_2)}\, i\omega_1\tau_1 a_k = \alpha'_{0k}. \tag{3.17}$$

Expressions for the friction force and for the heat flow due to the relative velocity have the following form (in order to obtain the total friction force the term $\mathbf{R}_1^{(0)}$ has also been added):

$$\begin{aligned} \mathbf{R}_{1u} &= \frac{m_1 n_1}{\tau_1}\left\{-\gamma_{uu}\mathbf{u}_{\parallel} - \mathbf{u}_{\perp} + \frac{1}{\Delta}(\beta'_{uu}x + \gamma'_{uu})\,\mathbf{u}_{\perp} \right. \\ &\qquad \left. - \frac{1}{\Delta}(\beta''_{uu}x + \gamma''_{uu})\,\tau_1\,[\boldsymbol{\omega}_1 \times \mathbf{u}]\right\}; \\ \mathbf{q}_{1u} &= -n_1 T_1\left\{\gamma_{Tu}\mathbf{u}_{\parallel} + \frac{1}{\Delta}(\beta'_{Tu}x + \gamma'_{Tu})\,\mathbf{u}_{\perp} \right. \\ &\qquad \left. - \frac{1}{\Delta}(\beta''_{Tu}x + \gamma''_{Tu})\,\tau_1\,[\boldsymbol{\omega}_1 \times \mathbf{u}]\right\}. \end{aligned} \tag{3.18}$$

The expression for Δ is here the same as in (3.16). The coefficients are given in the table. The coefficients β_{uT}, γ_{uT} coincide with the coefficients β_{Tu}, γ_{Tu}. This coincidence is, of course, not accidental. It amy be shown that it is a consequence of the principle of symmetry of kinetic coefficients (Onsager's principle).

The accuracy of the evaluation of the coefficients given in the table may be checked by calculating some of them in the next approximation. If in expansions (3.6) and (3.10) one keeps not two, but three terms, and if one breaks off in a corresponding manner the infinite systems (3.15) and (3.17), then for the coefficients γ one obtains the values $\gamma_{uT} = 0.7135$, $\gamma_{TT} = 3.1636$, $\gamma_{uu} = 0.5098$ (for $Z = 1$). Making use of the value of the coefficient γ_{uu} from the table one can obtain for the electrical conductivity of the plasma along the magnetic field the expression $\sigma_{\parallel} = 1.950\, n_1 e_1^2 \tau_1/m_1$. However, if one uses the more accurate value of the coefficient one obtains $\sigma_{\parallel} = 1.962\, n_1 e_1^2 \tau_1/m_1$. One may consider that the coefficients in the table have an accuracy of the order of 1%.

The equation for the correction to the ion distribution function is solved in the same manner as the one for the electrons. The infinite system of equations for the complex coefficients is analogous to the system (3.15) with only the single difference that cross matrix elements are absent. Therefore the expansion coefficients depend only on the parameter $\omega_2\tau_2$. The heat flow may be computed from formula (3.11) after replacing the subscript 1 by 2. As a result of this we obtain:

$$q_2 = -\frac{n_2 T_2 \tau_2}{m_2}\left\{3.906\, \nabla_{\parallel} T_2 + \frac{(2\omega_2^2\tau_2^2 + 2.645)\, \nabla_{\perp} T_2 - \left(\frac{5}{2}\,\omega_2^2\tau_2^2 + 4.65\right)\tau_2\,[\omega_2 \times \nabla T_2]}{\omega_2^4\tau_2^4 + 2.70\,\omega_2^2\tau_2^2 + 0.677}\right\}, \qquad (3.19)$$

where

$$\tau_2 = 3\sqrt{m_2}\, T_2^{3/2} / 4\sqrt{\pi}\,\lambda e_2^4 n_2, \quad \omega_2 = (e_2/m_2 c)\, \mathrm{H}. \qquad (3.20)$$

4. THE VISCOSITY TENSOR

In addition to the term proportional to the temperature gradient the right hand side of equation (2.14) also contains a term which depends on the space derivatives of the hydrodynamic velocity. The correction to the ion distribution function must also contain a corresponding term. The equation for this correction term has the form:

$$I(\Phi) + f^{(0)}\,[\mathbf{v}\omega]\,\nabla_v \Phi = -f^{(0)}\left(s_i s_k - \frac{s^2}{3}\,\delta_{ik}\right) w_{ik}, \qquad (4.1)$$

The subscript 2 (ions) has been omitted in equation (4.11) for the sake of brevity. The dimensionless variable **s** has been introduced in place of the velocity. The following abreviation is used below

$$s_{ik} = s_i s_k - (s^2/3)\delta_{ik}. \qquad (4.2)$$

We choose the z axis along the magnetic field. We decompose the tensor w_{ik} into a sum of three tensors $w_{ik} = w_{(0)ik} + w'_{(1)ik} + w'_{(2)ik}$ which in this system of coordinates have the components:

$$w_{(0)} = \begin{pmatrix} {}^1\!/_2\,(w_{xx} + w_{yy}) & 0 & 0 \\ 0 & {}^1\!/_2\,(w_{xx} + w_{yy}) & 0 \\ 0 & 0 & w_{zz} \end{pmatrix}; \qquad (4.3)$$

$$w'_{(1)} = \begin{pmatrix} {}^1\!/_2\,(w_{xx} - w_{yy}) & w_{xy} & 0 \\ w_{yx} & -{}^1\!/_2\,(w_{xx} - w_{yy}) & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad w'_{(2)} = \begin{pmatrix} 0 & 0 & w_{xz} \\ 0 & 0 & w_{yz} \\ w_{zx} & w_{zy} & 0 \end{pmatrix}. \qquad (4.4)$$

We also introduce tensors with the following components:

$$w''_{(1)} = \omega \begin{pmatrix} -2w_{xy} & w_{xx} - w_{yy} & 0 \\ w_{xx} - w_{yy} & 2w_{xy} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad w''_{(2)} = \omega \begin{pmatrix} 0 & 0 & -w_{yz} \\ 0 & 0 & w_{xz} \\ -w_{zy} & w_{zx} & 0 \end{pmatrix}. \qquad (4.5)$$

These tensors have been chosen in such a way that their traces vanish $w_{ii} = 0$, and the application of the operator $(\mathbf{v} \times \omega)\, \nabla_v$ transforms terms of the form $w'_{(1)ik}\, s_{ik}$, $w'_{(2)ik}\, s_{ik}$ into corresponding terms with double primes and conversely, and makes the term $w_{(0)ik}\, s_{ik}$ vanish.

The solution (4.1) may be sought in the form:

$$\Phi = \Phi_{(0)} + \Phi_{(1)} + \Phi_{(2)},$$

where

$$\Phi_{(0)} = -B_{(0)}(s^2)\, w_{(0)ik} s_{ik}, \quad \Phi_{(1)} = -(B'_{(1)} w'_{(1)ik} + B''_{(1)} w''_{(1)ik})\, s_{ik}, \quad \Phi_{(2)} = -(B'_{(2)} w''_{(2)ik} + B''_{(2)} w''_{(2)ik})\, s_{ik}. \tag{4.6}$$

All these terms may be calculated independently of each other.

It is convenient to expand the functions $B(s^2)$ into a series of Sonine polynomials of order $^5\!/_2$

$$B(s^2) = \tau \sum_{k=0}^{\infty} b_k L_k^{(^5/_2)}(s^2). \tag{4.7}$$

The contribution of the term $\Phi_{(0)}$ to the viscosity tensor is equal to

$$\pi_{(0)ik} = -nT\tau b_0 w_{(0)ik}\,. \tag{4.8}$$

Thus, in order to compute the viscosity it is sufficient to know only the zero order expansion coefficient, and the contribution to the viscosity tensor of $\Phi_{(1)}$ and $\Phi_{(2)}$ may be computed in a completely analogous manner. If one substitutes into (4.1) the expression (4.6) for $\Phi_{(0)}$ then one obtains for the function $B_{(0)}$ the equation

$$I(B_{(0)} s_{ik}) = f^{(0)} s_{ik}. \tag{4.9}$$

For the complex quantities $B_{(1)} = B'_{(1)} + 2i\omega B''_{(1)}$, $B_{(2)} = B'_{(2)} + i\omega B''_{(2)}$ the following equations are obtained:

$$I(B_{(1)} s_{ik}) + 2i\omega f^{(0)} B_{(1)} s_{ik} = f^{(0)} s_{ik}; \tag{4.10}$$

$$I(B_{(2)} s_{ik}) + i\omega f^{(0)} B_{(2)} s_{ik} = f^{(0)} s_{ik}. \tag{4.11}$$

Of Eqs. (4.9), (4.10), (4.11) it is sufficient to solve (4.11). Then by setting $\omega = 0$ we shall obtain the solution of (4.9), and by replacing ω by 2ω we shall obtain the solution of (4.10).

The method of solving (4.11) is the same as the method of solving the corresponding equations of Sec. 3. For the coefficients $b_{(2)}$ we obtain the infinite system of equations

$$\sum_{l=0}^{\infty} \beta_{kl} b_l + \frac{\Gamma(k + {^7\!/_2})}{k!\,\Gamma({^7\!/_2})} i\omega\tau b_k = \delta_{0k}. \tag{4.12}$$

The matrix elements β_{kl} are calculated in the Appendix. The coefficient b_0 may be obtained by breaking off after the first two terms the expansion (4.7) and corresponding to this also the system (4.12)

$$b_{(2)0} = b' + i\omega\tau b'' = \frac{({^6\!/_5}\,\omega^2\tau^2 + 2.23) - i\omega\tau(\omega^2\tau^2 + 2.38)}{\omega^4\tau^4 + 4.03\,\omega^2\tau^2 + 2.33}\,. \tag{4.13}$$

Collecting all the terms we obtain for the components of the viscosity tensor for the ions the expressions:

$$\begin{gathered}
\pi_{zz} = -0.96\, n_2 T_2 \tau_2 w_{zz}, \\
\pi_{xx} = n_2 T_2 \tau_2 \left\{0.48\ w_{zz} - \frac{1}{2} b'(2\omega)(w_{xx} - w_{yy}) - 2\omega_2\tau_2 b''(2\omega)\, w_{xy}\right\}, \\
\pi_{yy} = n_2 T_2 \tau_2 \left\{0.48\ w_{zz} + \frac{1}{2} b'(2\omega)(w_{xx} - w_{yy}) + 2\omega_2\tau_2 b''(2\omega)\, w_{xy}\right\}, \\
\pi_{xy} = \pi_{yx} = -n_2 T_2 \tau_2 \{b'(2\omega)\, w_{xy} - \omega_2\tau_2 b''(2\omega)(w_{xx} - w_{yy})\}, \\
\pi_{xz} = \pi_{zx} = -n_2 T_2 \tau_2 \{b'(\omega) w_{xy} + w_2\tau_2\, b''(\omega) w_{yz}\}, \\
\pi_{yz} = \pi_{zy} = -n_2 T_2 \tau_2 \{b'(\omega) w_{yz} - \omega_2\tau_2\, b''(\omega) w_{xz}\}.
\end{gathered} \tag{4.14}$$

Expressions (4.14) differ from the corresponding results given (without proof) by Chapman and Cowling[11] in having more accurate coefficients. The equation which determines the electronic viscosity is analogous to the equation for the ions, and is solved in the same way. The system of equations for the complex coefficients b_k has the form:

$$\sum_{l=0}^{\infty} (\beta_{kl} + \beta'_{kl})\, b_l + \frac{\Gamma(k + 7/2)}{k!\,\Gamma(7/2)} i\omega_1\tau_1 b_k = \delta_{0k}. \tag{4.15}$$

The matrix elements for electron-electron and electron-ion collisions are calculated in the Appendix. The coefficients b depend on the effective charge of the ions Z. For Z = 1

$$b' + i\omega_1\tau_1 b'' = \frac{(2.05\omega_1^2\tau_1^2 + 8.50) - i\omega_1\tau_1(\omega_1^2\tau_1^2 + 7.91)}{\omega_1^4\tau_1^4 + 13.8\omega_1^2\tau_1^2 + 11.6}. \tag{4.16}$$

The viscosity tensor for the electrons as well as for the ions has the form (4.14), where one needs only to replace the subscript 2 by 1 and to take the values of the coefficients in accordance with (4.16).

In conclusion I wish to express my deep gratitude to B. I. Davydov for suggesting the problem and for his considerable aid in selecting the method for its solution. I also express my thanks to M. A. Leontovich and to G. I. Budker for numerous useful discussions.

APPENDIX

Calculation of Matrix Elements

In the course of calculations in Secs. 3 and 4 it is necessary to evaluate integrals of the form:

$$M_{pq}^{\alpha\beta} = \int \Psi_p S_{\alpha\beta}(f_\alpha^{(0)}\Phi_q, f_\beta^{(0)})\,d\mathbf{v}, \quad N_{pq}^{\alpha\beta} = \int \Psi_p S_{\alpha\beta}(f_\alpha^{(0)}, f_\beta^{(0)}\Phi_q)\,d\mathbf{v}. \tag{A.1}$$

The role of Ψ_p and Φ_q is played by functions of the form

$$L_p^{(3/2)}\left(\frac{mv^2}{2T}\right)v_i, \quad L_p^{(5/2)}\left(\frac{mv^2}{2T}\right)\left(v_i v_k - \frac{v^2}{3}\delta_{ik}\right),$$

where $L_p^{(m)}$ are Sonine polynomials (generalized Laguerre polynomials). These polynomials have the following generating functions:

$$(1-\xi)^{-5/2}\exp\left(-\frac{t\xi}{1-\xi}\right) = \sum_{p=0}^{\infty}\xi^p L_p^{(3/2)}(t), \quad (1-\xi)^{-7/2}\exp\left(-\frac{t\xi}{1-\xi}\right) = \sum_{p=0}^{\infty}\xi^p L_p^{(5/2)}(t). \tag{A.2}$$

We shall call integrals of the type (A.1) matrix elements. Instead of calculating these matrix elements individually for each pair of values of p, q, it is convenient to calculate the matrix element of the generating functions (A.2). Then by expanding it in powers of ξ, η we shall obtain the desired matrix elements in the form of coefficients of $\xi^p\eta^q$. For example, for polynomials of order $3/2$:

$$M^{12}\left(\frac{3}{2}\right) = \sum_{p,\,q=0}^{\infty}\xi^p\eta^q M_{pq}^{12}\left(\frac{3}{2}\right). \tag{A.3}$$

All the required integrals are calculated by the same method, by integration by parts and change of variables $v_i = c_i + \gamma u_i$, $v'_i = c_i + \gamma' u_i$ where the coefficients γ, γ' are so chosen that the exponent contains no terms of the form $c_i u_i$ and that $v_i - v'_i = u_i$. As a result we obtain

$$M^{12}\left(\frac{3}{2}\right) = \frac{8\sqrt{\pi}\lambda e_1^2 e_2^2 n_1 n_2}{m_1^2}(1-\xi\eta)^{-3/2}(1-\xi)^{-1}(1-\eta)^{-1}\left(\frac{\beta_1\beta_2(1+x+y)}{B}\right)^{1/2}\left\{1 - \frac{(x+y)\beta_1}{B} + \frac{5xy\beta_1^2}{B^2} + \frac{2xy\beta_1\beta_2}{B^2(1+x+y)}\right.$$
$$\left. - \beta_1\left(1-\frac{T_1}{T_2}\right)\left(\frac{1}{B} - \frac{5x\beta_1}{B^2} - \frac{2x\beta_2}{B^2(1+x+y)}\right)\right\}, \quad N^{12}\left(\frac{3}{2}\right) = -\frac{8\sqrt{\pi}\lambda e_1^2 e_2^2 n_1 n_2}{m_1 m_2}(1-\xi)^{-1}(1-\eta)^{-1}\left(\frac{\beta_1\beta_2(1+x)(1+y)}{A}\right)^{1/2}$$
$$\times\left\{1 - \frac{x\beta_1 + y\beta_2}{A} + \frac{3xy\beta_1\beta_2}{A^2} + \beta_2\left(\frac{T_2}{T_1} - 1\right)\left(\frac{1}{A} - \frac{3x\beta_1}{A^2}\right)\right\}. \tag{A.4}$$

Here we have introduced the notation

$$\beta_1 = m_1/2T_1, \quad \beta_2 = m_2/2T_2; \tag{A.5}$$

$$x = \frac{\xi}{1-\xi}; \quad y = \frac{\eta}{1-\eta}, \quad A = \beta_1(1+x) + \beta_2(1+y), \quad B = \beta_1(1+x+y) + \beta_2. \tag{A.6}$$

With the aid of (A.4) it is possible to obtain expressions for all the required matrix elements. For collisions between electrons and ions we have, taking into account that $\beta_1 \ll \beta_2$:

$$\sum_{p,\,q=0}^{\infty}\xi^p\eta^q M_{pq}^{12}\left(\frac{3}{2}\right) = \frac{3n_1}{\tau_1}\frac{T_1}{m_1}(1-\xi\eta)^{-1}(1-\xi)^{-3/2}(1-\eta)^{-3/2}, \quad \sum_{p,\,q=0}^{\infty}\xi^p\eta^q N_{pq}^{12}\left(\frac{3}{2}\right) = -\frac{3n_1}{\tau_1}\frac{T_2}{m_2}(1-\xi)^{-3/2}. \tag{A.7}$$

For collisions between ions and electrons we have after interchanging subscripts 1 and 2 and taking into account that $\beta_1 \ll \beta_2$

$$\sum_{p,\,q=0}^{\infty} \xi^p \eta^q M_{pq}^{21}\left(\frac{3}{2}\right) = \frac{3n_1}{\tau_1}\frac{T_2}{m_1}(1-\xi\eta)^{-7/2}\left\{1-5\xi\left(\frac{T_1}{T_2}-1\right)+4\xi\eta\right\}, \quad \sum_{p,\,q=0}^{\infty} \xi^p \eta^q N_{pq}^{21}\left(\frac{3}{2}\right) = -\frac{3n_1}{\tau_1}\frac{T_1}{m_2}(1-\eta)^{-5/2}. \tag{A.8}$$

For collisions of identical particles, for example of ions with ions, we have to calculate matrix elements of the form: $H_{pq}^{22}(3/2) = M_{pq}^{22}(3/2) + N_{pq}^{22}(3/2)$,

$$\sum_{p,\,q=0}^{\infty} \xi^p \eta^q H_{pq}^{22}\left(\frac{3}{2}\right) = \frac{3n_2}{\tau_2}\frac{2T_2}{m_2}\xi\eta(1-\xi\eta)^{-2}\left(1-\frac{\xi+\eta}{2}\right)^{5/2}\left\{1-\frac{\xi+\eta}{2}-\frac{\xi\eta}{8}+\frac{\xi\eta(\xi+\eta)}{4}+\frac{3}{8}\xi^2\eta^2\right\}, \tag{A.9}$$

where $\tau_2 \equiv \tau_{22} = 3\sqrt{m_2}\,T_2^{3/2}/4\sqrt{\pi}\,\lambda e_2^4 n_2$. The corresponding expression for the electrons may be obtained from (A.9) by replacing the subscript 2 by 1, with τ_2 being replaced by $\tau_{11} = \sqrt{2}\,Z\tau_1$. The values of the coefficients α_{pq} used in Sec. 3 may be obtained from (A.9). For the ions:

$$\alpha_{pq} = \frac{4}{5}\begin{pmatrix} 0 & 0 & 0 & 0 & . \\ 0 & 1 & \frac{3}{4} & \frac{15}{32} & . \\ 0 & \frac{3}{4} & \frac{45}{16} & \frac{309}{128} & . \\ 0 & \frac{15}{32} & \frac{309}{128} & \frac{6425}{1024} & . \\ . & . & . & . & . \end{pmatrix}. \tag{A.10}$$

The coefficients α_{pq} for the electrons are smaller by a factor $\sqrt{2}\,Z$.

The values of the coefficients α'_{pq} may be obtained from (A.7)

$$\alpha_{pq} = \frac{2}{5}\begin{pmatrix} 1 & \frac{3}{2} & \frac{15}{8} & \frac{35}{16} & . \\ \frac{3}{2} & \frac{13}{4} & \frac{69}{16} & \frac{165}{32} & . \\ \frac{15}{8} & \frac{69}{16} & \frac{433}{64} & \frac{1077}{128} & . \\ \frac{35}{16} & \frac{165}{32} & \frac{1077}{128} & \frac{2954}{256} & . \\ . & . & . & . & . \end{pmatrix} \tag{A.11}$$

Matrix elements with Sonine polynomials of order $5/2$ which are required for calculating the viscosity tensor may be found by the same method which was used in the case of polynomials of order $3/2$. We require only

$$H_{pq}^{22}\left(\frac{5}{2}\right) = \int L_p^{(5/2)}(s^2)\, s_{ik} I_2\left(L_q^{(5/2)}(s^2)\, s_{ik}\right) dv, \; H_{pq}^{11}\left(\frac{5}{2}\right), \quad M_{pq}^{12}\left(\frac{5}{2}\right) = \int L_p^{(5/2)}(s'^2)\, s_{ik} I'_{12}\left(L_q^{(5/2)}(s^2)\, s_{ik}\right) dv. \tag{A.12}$$

They are equal to

$$\sum_{p,\,q=0}^{\infty} \xi^p \eta^q M_{pq}^{12}\left(\frac{5}{2}\right) = \frac{3n_1}{\tau_1}(1-\xi)^{-3/2}(1-\eta)^{-3/2}(1-\xi\eta)^2, \quad \sum_{p,\,q=0}^{\infty} \xi^p \eta^q H_{pq}^{2}\left(\frac{5}{2}\right) = \frac{3n_2}{\tau_2}\left(1-\frac{\xi+\eta}{2}\right)^{-5/2}(1-\xi\eta)^{-1}\left\{1-\frac{\xi+\eta}{2}\right.$$
$$\left. + \xi\eta\left[\frac{7}{3}+\frac{2\xi\eta-(\xi+\eta)/2-\xi\eta(\xi+\eta)/12}{1-\xi\eta}+\frac{\xi\eta}{3(1-\xi\eta)^2}+\frac{5}{16}\frac{\xi\eta}{1-(\xi+\eta)/2}\right]\right\}. \tag{A.13}$$

The expressions for H_{pq}^{11} are obtained from H_{pq}^{22} by replacing 2 by 1 and τ_2 by $\tau_{11} = \sqrt{2}\,Z\tau_1$. The coefficients β_{pq}, β'_{pq} are equal to:

$$\beta_{pq} = \frac{6}{5}\begin{pmatrix} 1 & \frac{3}{4} & . \\ \frac{3}{4} & \frac{205}{48} & . \\ . & . & . \end{pmatrix}, \quad \beta'_{pq} = \frac{6}{5}\begin{pmatrix} 1 & \frac{3}{2} & . \\ \frac{3}{2} & \frac{17}{4} & . \\ . & . & . \end{pmatrix} \tag{A.14}$$

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ANOMALOUS EQUATIONS FOR SPIN 1/2 PARTICLES

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Irreducible relativistic wave equations different from the Dirac equations are derived for spin ½ particles. The particles described by these equations may have one or more proper masses and the corresponding fields have a positive definite charge density. The existence of such equations is not incompatible with the well known proof of uniqueness of the Dirac equations.

ATTEMPTS to construct wave equations of the type

$$(\beta^k \partial_k - i\varkappa)\varphi = 0 \tag{1}$$

but different from the Dirac equations have been unsuccessful. The only exception is the set of equations due to Petras.[1] But these equations are not investigated in details in his paper, and it is therefore not clear whether they meet all of the physical requirements.

The known proofs by Wild[2] and by Gel'fand and Iaglom[3] on the uniqueness of the Dirac equations seemed to imply that anomalous equations for spin ½ particles do not exist, and that the equations of Petras[1] are physically unacceptable.

The proofs of the uniqueness of the Dirac equation are, however, not general. They rely on the assumption that the equation for spin ½ particles can be derived only with the representation of the total Lorentz group for the maximum value of spin ½.

In this paper we start from the representation of D_R for the maximum value of spin $3/2$ and we show that there exist anomalous equations for spin ½ particles, the simplest of them being the Petras equations.

1. RELATIVISTIC FORM OF ANOMALOUS WAVE EQUATIONS

If the equations (1) are covariant, the matrices β_k have to satisfy the known relationships

$$[\beta_k I_{lm}] = g_{kl}\beta_m - g_{km}\beta_l; \tag{2a}$$

$$[I_{kl} I_{mn}] = -g_{km} I_{ln} + g_{lm} I_{kn} + g_{kn} I_{lm} - g_{ln} I_{km}; \tag{2b}$$

$$\beta_j Z = Z\beta^j, \tag{2c}$$

where $I_{k\ell}$ represent infinitesimal rotations and Z is the matrix of space reflection.

If the representation of the total Lorentz group D_R is known, Eqs. (2) can be used to find the form of the matrix β_0. The remaining matrices β_ν $(\nu = 1, 2, 3)$ can be determined from (2a).

It is known that there are three non-equivalent irreducible representations of the operators $I_{k\ell}$ and Z for the maximum value of spin $3/2$, with 12, 8 and 4 rows respectively, which we shall denote by τ_3, τ_2 and τ_1.

In order to construct the matrix β_0, we can use the general representation of the elements $I_{k\ell}$ and Z

$$I_{kl} = aI_{kl}^{(\tau_3)} + bI_{kl}^{(\tau_2)} + cI_{kl}^{(\tau_1)}; \quad Z = aZ^{(\tau_3)} + bZ^{(\tau_2)} + cZ^{(\tau_1)},$$

where a, b and c determine the number of irreducible representations of the same kind. However, in the following discussion, we will restrict ourselves to cases with $a = 1$, $b = 0$, $c = 1, 2, 3 \ldots$; these cases are the simplest because the existing field has then a positive definite charge density; in the general case the matrices β_k are either irreducible or not equal to the Dirac matrices.

We will denote the matrix β_0 constructed with these representations by

$$\beta_0 = \beta_0(\tau_3, c\tau_1).$$

It has been shown[4] that the matrix β_0 can be written as a direct product of two matrices

$$\beta_0 = \gamma_0 \times \alpha(\tau_3, c\tau_1), \tag{3}$$

where γ_0 is a Dirac matrix. The same reference shows the form of the matrix α ($\alpha = U\alpha_0U^+$, α_0 is a matrix from Ref. 4) to be

$$\alpha \equiv \alpha(\tau_3, c\tau_1) = \left\| \begin{array}{cccc} 2l & & & \\ & 2l & & \\ & & -l & R \\ & & S & K \end{array} \right\|$$

R is a row matrix with elements r_μ, $\mu = 1, 2, \ldots, c$; S is a column matrix with elements s_μ, $\mu = 1, 2, \ldots, c$.

Some of the coefficients of the matrix α can be assumed equal to zero.

If we set

$$l = 0;\ k_{\mu\nu} = 0,\ \ \mu \neq \nu;\ k_{\mu\mu} = k_\mu \neq 0;\ k_\mu \neq \pm k_\nu,\ \ \mu \neq \nu;\ r_\mu \neq 0;\ s_\mu \neq 0,$$

it is easy to verify that the matrices β_k remain irreducible and that the matrix α will have the simple form

$$\alpha \equiv \alpha(\tau_3, c\tau_1) = \left\| \begin{array}{ccccccc} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & r_1 & r_2 & \ldots & r_c \\ & & s_1 & k_1 & & & \\ & & s_2 & & k_2 & & \\ & & \vdots & & & \ddots & \\ & & s_c & & & & k_c \end{array} \right\|$$

This matrix has in general c + 1 different eigenvalues λ_μ ($\mu = 1, 2, \ldots, c$) different from zero

For each eigenvalue λ_μ there are four independent solutions of the wave equation (1); these solutions correspond to the two signs of the energy $\pm E$ and to the two directions of spin $\pm 1/2$. The corresponding wave equations describe therefore a spin 1/2 particle which has, in the general case, c + 1 different mass values.

It follows that, in principle, there can exist spin 1/2 relativistic wave equations which differ from the Dirac equation. We shall call them anomalous because the spin 3/2 state never occurs although we start from the representation of D_R for the maximum value of spin 3/2. The spin 3/2 state is excluded by the condition $\ell = 0$.

It is easy to find the anomalous wave equations in explicit form. We have in the spin-tensor form

$$-\sum_{\mu=1}^{c} r_\mu \left(2\partial^k\chi^{(\mu)} + \gamma^l\gamma^k\partial_l\chi^{(\mu)}\right) = \varkappa B^k, \quad {}^1\!/_2 s_\mu \partial_k B^k - ik_\mu\gamma^k\partial_k\chi^{(\mu)} = \varkappa\chi^{(\mu)}, \tag{4}$$

where $\chi^{(\mu)}$ is the Dirac spinor, B^k is a spin-vector introduced by Rarita and Schwinger,[5] and γ^k are the usual Dirac matrices.

2. ANOMALOUS EQUATIONS FOR SPIN 1/2 PARTICLES WITH A SINGLE MASS EIGENVALUE

In the general case, i.e., for arbitrary values of the coefficients r_μ, s_μ and k_μ, the charge density of the field described by Eq. (4) is not positive definite. However, one can choose the relations between the coefficients of the matrix α such as to make the charge density positive definite.

The problem is treated best in the rest system of the particle. If Eq. (1) is solved in this system, letting

$$\partial_0 \to iE = i\varkappa/\lambda,$$

the following equation for the amplitude φ is obtained

$$(\beta^0 - \lambda)\,\varphi = 0. \tag{5}$$

The charge density is determined from the expression

$$\rho = \varphi^{+}\Lambda\beta^0\varphi, \tag{6}$$

where the matrix Λ has to satisfy the condition

$$\Lambda\beta_k = \beta_k^{+}\Lambda.$$

The matrix Λ can be expressed as a direct product of the Dirac matrix γ^0 and the matrix $\eta = \mathrm{diag}\,\|1, 1, 1, \epsilon_1, \epsilon_2, \ldots \epsilon_c\|$, $(\epsilon_\mu = \pm 1)$;

$$\Lambda = \gamma_0\eta. \tag{7}$$

Substituting the direct products (3) and (7) into (5) and (6), we get

$$(\alpha - \lambda)\,\varphi_{1,3} = 0, \quad (\alpha + \lambda)\,\varphi_{2,4} = 0; \tag{5'}$$

$$\rho = \sum_{r=1}^{4} \varphi_r^{+}\eta\alpha\varphi_r, \tag{6'}$$

where φ_r are the components of φ. The charge density will obviously be positive definite if we can show that one of the terms of the sum (6′), e.g., the term $\varphi_1^{+}\eta\alpha\varphi_1$, is positive. The condition $\rho > 0$ can therefore be replaced by the condition

$$\varphi_1^{+}\eta\alpha\varphi_1 = \lambda\varphi_1^{+}\eta\varphi_1 > 0, \tag{8}$$

in which the amplitude φ_1 has five components φ_μ.

The case $\alpha = \alpha(\tau_3, \tau_1)$, i.e., $c = 1$, does not yield a relationship between the coefficients of the matrix α such that the relationship (8) be satisfied. We therefore turn to the case $\alpha = \alpha(\tau_3, 2\tau_1)$ i.e., $c = 2$.

If

$$r_1^2 = \varepsilon_1 k_1^2 k_2/(k_1 - k_2), \; r_2^2 = \varepsilon_2 k_2^2(-k_1)/(k_1 - k_2), \quad k_1 > k_2, \quad s_1 = \varepsilon_1 r_1, \quad s_2 = \varepsilon_2 r_2, \tag{9}$$

the matrix α will have a single eigenvalue λ different from zero: $\lambda = k_1 + k_2$. This means that the particle will have a single mass $m = \kappa/|\lambda|$, or $m = \kappa$ if we choose $|\lambda| = 1$.

If all the dependent components are excluded from (8), we get

$$\lambda\,(\lambda^2/k_1k_2)\,\overset{*}{\psi}_3\psi_3$$

which will obviously be positive definite if $\lambda > 0$, $k_2 > 0$ or $\lambda < 0$, $k_1 > 0 > k_2$. In the first case we have $\epsilon_1 = -\epsilon_2 = 1$; in the other case $\epsilon_1 = \epsilon_2 = -1$. In both cases the matrix β_0 is not Hermitian and it cannot be diagonalized because it has a multiple zero value. It is easy to show that a similarity transformation which would transform the case $\epsilon_1 = -\epsilon_2$ into the case $\epsilon_1 = \epsilon_2$ does not exist; it can be shown that all the other cases can be obtained through a similarity transformation.

The corresponding matrix β_0, which has 20 rows, satisfies the equation

$$\beta_0^2(\beta_0^2 - 1) = 0.$$

All the coefficients of α cannot be determined by the conditions (9) and the mass of the particle. One of the coefficients is independent and the wave function contains therefore a free parameter.

3. ANOMALOUS EQUATIONS FOR SPIN 1/2 PARTICLES WITH SEVERAL MASS EIGENVALUES

Let us consider the case $\alpha = \alpha(\tau_3, c\tau_1)$ with $c = 3$. We see that the coefficients of the matrix can be chosen in three independent ways such that the particle has two different masses $m_1 = \kappa/|\lambda_1|$, $m_2 = \kappa/|\lambda_2|$ and that the corresponding field has a positive definite charge density (i.e., the charge density has the same spin for both mass states, and is positive).

We will show the relationships which have to be satisfied by the coefficients of the matrix α in only one of the possible cases:

$$\begin{gathered} r_1^2 = k_1^2 (k_1 - \lambda_1)(k_1 - \lambda_2) / (k_3 - k_1)(k_1 - k_2), \; r_2^2 = -k_2^2 (k_2 - \lambda_1)(k_2 - \lambda_2) / (k_1 - k_2)(k_2 - k_3), \\ r_3^2 = k_3^2 (k_3 - \lambda_1)(k_3 - \lambda_2) / (k_3 - k_1)(k_2 - k_3), \quad s_1 = r_1, \quad s_2 = -r_2, \quad s_3 = r_3, \\ \varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = 1, \quad \lambda_1 > k_1 > k_2 > k_3 > \lambda_2, \quad \lambda_2 < \lambda_1/2, \quad \lambda_1 + \lambda_2 = k_1 + k_2 + k_3. \end{gathered} \tag{10}$$

This case is interesting because the masses of the particles are not independent—in the remaining two cases a similar restriction does not exist.

As it can be seen from Eq. (10), the wave equation will contain two free parameters.

The number of possible choices of the coefficients of the matrix increasing with the increase of c, there exist, for each matrix $\alpha = \alpha(\tau_3, c\tau_1)$, $c \geq 1$, physically acceptable anomalous equations. In general, they describe spin ½ particles with $c - 1$ different masses.

CONCLUSION

The anomalous wave equations for spin ½ particles can be generated from the Lagrange function because for each case there is a matrix Λ (7). The electromagnetic interaction can be introduced in the anomalous equations the usual way, i.e., by making the substitution

$$\partial_k \rightarrow \partial_k - ieA_k$$

without destroying their simultaneity. An obvious proof for the case $c = 2$ has been given in Ref. 1. Similarly, the introduction of other kinds of interactions does not present any difficulty; indeed the Lagrangian of the interaction contains, for a given kind of coupling (e.g., pseudoscalar), several independent invariants; we have therefore at our disposal several coupling constants and we can find some relationships between them such that the number of additional (initial) conditions contained in the system (4) is conserved when the interaction is introduced.

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ON THE IMPROVEMENT OF PERTURBATION THEORY FORMULAS

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The n-fold Compton effect and high-energy scattering of electrons and positrons by electrons are investigated, taking into account vacuum polarization diagrams composed of the simplest closed loops.

Bogoliubov and Shirkov[1] have shown that it is possible to improve the perturbation theory formulas, owing to the existence of the renormalization group. In the present paper the mentioned method is applied to the investigation of the Compton effect and high-energy scattering of electrons and positrons by electrons, taking into account vacuum polarization.

As known the renormalized Green's functions admit a group of finite transformations

$$G_1 \to G_2 = z_2 G_1;\quad \Gamma_1 \to \Gamma_2 = z^{-1}\Gamma_1;\quad z_1 = z_2;\quad D_1 \to D_2 = z_3 D_1;\quad e_1 \to e_2 = z_3^{-1/2} e_1, \tag{1}$$

meaning that the same results are obtained when either $(G_1, D_1, \Gamma_1, e_1)$ or $(G_2, D_2, \Gamma_2, e_2)$ are used. The transformations (1) should be supplemented by the transformation of the generalized vertex part

$$\Gamma_1^{(n)} \to \Gamma_2^{(n)} = z_1^{-1}\Gamma_1^{(n)}, \text{ where } \Gamma^{(n)} = G^{-1}\frac{\delta^n G}{\delta(-eA)^n}G^{-1}, \tag{2}$$

and by the Green's function for two electrons

$$G_1^{(2)} \to G_2^{(2)} = z_2^2 G_1^2. \tag{3}$$

Using transformations (1) to (3) one obtains a functional equation for the Green's function, in a manner analogous to the one used by Bogoliubov and Shirkov.

We write the photon Green's function in the following form

$$D_{\mu\nu}(k) = -k^{-2}(g^{\mu\nu} - k_\mu k_\nu/k^2)\, d(k^2). \tag{4}$$

In the present paper we restrict ourselves to the following values of the function $d(k^2)$:

$$d(k^2) = \left[1 - \frac{e^2}{3\pi}\ln\frac{k^2}{m^2}\right]^{-1}. \tag{5}$$

This means that we are taking into account the simplest closed loops.

The generalized Green's function for the Compton effect in the high-energy region can be presented in the form

$$\Gamma_{\mu\nu} = \left(\gamma_\mu \frac{\hat{p}_1 + \hat{l}_1 + m}{(p_1 + l_1)^2 - m^2}\gamma_\nu + \gamma_\nu \frac{\hat{p}_1 - \hat{l}_2 + m}{(p_1 - l_2)^2 - m^2}\gamma_\mu\right)\Gamma((p_1 l_1), (p_2 l_2)), \tag{6}$$

where p_1 is the initial 4-momentum of the electron and ℓ_1 and ℓ_2 are the 4-momenta of the absorbed and emitted photons. Accordingly, the transition probability will be written in the form $d\sigma = d\sigma_0 W$ where $d\sigma_0$ is the effective cross section in the zeroth order perturbation theory.

Using the arguments analogous to the ones found in Ref. 1, we obtain the following equations for the function Γ:

$$\ln\frac{\Gamma(x, y, u, e^2)}{\Gamma(x_0, y, u, e^2)} = \int_{x_0}^{x}\frac{dt}{t}\left[\frac{\partial}{\partial\xi}\ln\Gamma\left(\xi, \frac{y}{t}, \frac{u}{t}, e^2 d(t, u, e^2)\right)\right]_{\xi=1} \tag{7}$$

$$\ln\frac{\Gamma(x_0, y, u, e^2)}{\Gamma(x_0, y_0, u, e^2)} = \int_{y_0}^{y}\frac{dt}{t}\left[\frac{\partial}{\partial\xi}\ln\Gamma\left(\frac{x_0}{t}, \xi, \frac{u}{t}, e^2 d(t, u, e^2)\right)\right]_{\xi=1}, \tag{8}$$

where $x = (p_1 l_1)/\lambda_2^2, \quad y = (p_1 l_2)/\lambda_2^2, \quad u = m^2/\lambda_2^2.$

Equations, analogous to (7) and (8), hold also for the correction to the probability W.

The radiative corrections for the Compton effect are, at high energies, equal to[2]

$$W = 1 - \frac{e^2}{\pi}\left(\frac{3}{2}\ln^2\frac{(p_1 l_1)}{m^2} - \frac{1}{2}\ln^2\frac{(p_1 l_1)}{(p_1 l_2)}\right) \tag{9}$$

for $(p_1 l_1) \gg m^2$, $(p_1 l_1) \gg (p_1 l_2)$.

We will also assume that $\Delta\epsilon = m$. When a hard photon k is emitted whose 4-momentum satisfies the conditions

$$(p_1 k) \ll (p_1 p_2), \quad (p_2 k) \ll (p_1 p_2), \quad m^{-2}(p_1 p_2)(p_1 k) > (p_2 k) > (p_1 k)\ m^2/(p_1 p_2), \tag{10}$$

the expression for the probability, according to Abrikosov, should be multiplied by the factor

$$W_1 = \frac{e^2}{\pi}\left(\frac{3}{2}\ln^2\frac{(p_1 l_1)}{m^2} - \frac{1}{2}\ln^2\frac{(p_1 l_1)}{(p_1 l_2)}\right). \tag{11}$$

Using the perturbation-theory formulas (9) and (11) and Eqs. (7) and (8), we obtain the following expression for the n-fold Compton effect with emission of an arbitrary number of photons:

$$d\sigma = d\sigma_0 e^{-\overline{N}}(\overline{N})^n/n!; \tag{12}$$

$$\overline{N} = \frac{3}{2}\ln\frac{(p_1 l_1)}{m^2}\ln d\left(\frac{(p_1 l_2)}{m^2}\right) + \frac{3}{2}\ln\frac{(p_1 l_2)}{m^2}\ln d\left(\frac{(p_1 l_1)}{m^2}\right) + 3\ln\frac{(p_1 l_2)}{m^2} - \frac{9\pi}{e^2}\ln d\left(\frac{(p_1 l_2)}{m^2}\right) - 6\ln\frac{(p_1 l_1)}{m^2} + \frac{18\pi}{e^2}\ln d\left(\frac{(p_1 l_1)}{m^2}\right). \tag{13}$$

This expression holds for

$$e^2 d(m^{-2}(p_1 l_1)) \ll 1, \quad e^2 d(m^{-2}(p_1 l_2)) \ll 1. \tag{14}$$

If the conditions

$$\frac{e^2}{3\pi}\ln\frac{(p_1 l_1)}{m^2} \ll 1, \quad \frac{e^2}{3\pi}\ln\frac{(p_1 l_2)}{m^2} \ll 1, \tag{15}$$

are satisfied, we obtain $\overline{N} \to \overline{n}$, where, according to Ref. 3:

$$\overline{n} = \frac{e^2}{\pi}\left(\frac{3}{2}\ln^2\frac{(p_1 l_1)}{m^2} - \frac{1}{2}\ln^2\frac{(p_1 l_1)}{(p_1 l_2)}\right). \tag{16}$$

Making use of the perturbation-theory formula for the Compton-effect radiative corrections[2]

$$W_2 = 1 + \frac{e^2}{2\pi}\ln^2\frac{(p_1 l_1)}{m^2}, \quad (p_1 l_1) \approx (p_1 l_2) \gg m^2, \tag{17}$$

we obtain the following expression for the index of refraction at high frequencies:

$$1 - n^2 = \frac{4\pi N e^2}{m\omega^2}\exp\left\{-\frac{3}{2}\ln\frac{\omega}{m} + \frac{9\pi}{2e^2}\ln d\left(\frac{\omega}{m}\right)\right\}, \quad e^2 d\left(\frac{\omega}{m}\right) \ll 1, \tag{18}$$

where ω is the frequency of the primary photon in the laboratory system. For $(e^2/3\pi)\ln(\omega/m) \ll 1$, the known result of Abrikosov[3] is obtained.

Let us consider electron-electron scattering. For $p \gg m$ (p is the momentum of the electron in the center-of-mass system) and for scattering angles θ close either to zero or to π and satisfying respectively the conditions $p\sin(\theta/2) \sim 1$ or $p\cos(\theta/2) \sim 1$, the differential cross section, according to Akhizer and Polovin,[4] is given by

$$d\sigma = d\sigma_0\left(1 - 1.74\frac{e^2}{\pi}\ln\frac{(p_1 p_2)}{m^2}\right), \tag{19}$$

where p_1 and p_2 are the 4-momenta of the initial state of the electrons. Applying Eq. (7) to this expression, we obtain

$$d\sigma = d\sigma_0[d(m^{-2}(p_1 p_2))]^{-5.22} \quad \text{for } e^2 d(m^{-2}(p_1 p_2)) \ll 1. \tag{20}$$

In the case of high-energy positron-electron scattering at an angle $m/E \ll \pi - \theta \ll \pi/2$ in the center-of-mass system, we have[3]

$$d\sigma = d\sigma_0\left(1 + \frac{e^2}{2\pi}\ln^2\frac{(p_1q_1)}{(p_1q_2)}\right), \quad (p_1q_1) \gg (p_1q_2), \tag{21}$$

where p_1 and q_1 are the 4-momenta of the electron in the initial and final state, and q_2 is the 4-momentum of the positron in the final state. Applying Eqs. (7) and (8) to this expression we find

$$d\sigma = d\sigma_0 \exp\left\{-\frac{3}{2}\ln\frac{(p_1q_1)}{m^2}\ln d\left(\frac{(p_1q_2)}{m^2}\right) - \frac{3}{2}\ln\frac{(p_1q_2)}{m^2}\ln d\left(\frac{(p_1q_1)}{m^2}\right) - 3\ln\frac{(p_1q_1)}{m^2} + \frac{9\pi}{e^2}\ln d\left(\frac{(p_1q_1)}{m^2}\right)\right.$$

$$\left. - 3\ln\frac{(p_1q_2)}{m^2} + \frac{9\pi}{e^2}\ln d\left(\frac{(p_1q_2)}{m^2}\right)\right\}, \tag{22}$$

for $e^2 d(m^{-2}(p_1q_1)) \ll 1$, $e^2 d(m^{-2}(p_1p_2)) \ll 1$.

If we assume that

$$\frac{e^2}{3\pi}\ln\frac{(p_1q_1)}{m^2} \ll 1, \quad \frac{e^2}{3\pi}\ln\frac{(p_1q_2)}{m^2} \ll 1,$$

we obtain again the known result of Abrikosov.

In conclusion, I express my gratitude to D. V. Shirkov for making available to me, prior to publication, his paper prepared in collaboration with V. Z. Blank and for a series of interesting remarks.

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SOVIET PHYSICS JETP VOLUME 6 (33) NUMBER 2 FEBRUARY, 1958

ON THE PRODUCTION OF π MESONS BY PROTONS ON NUCLEI OF VARIOUS ELEMENTS

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The problem considered is that of the dependence of the probability of π meson production in proton-nucleus collisions on the number A of nucleons in the nucleus. The deviation from the $A^{2/3}$ law is explained by the effective attenuation of the proton beam penetrating into the nucleus.

THE dependence of the cross-section for production of π^+ mesons on the atomic number in collisions of protons with nuclei of various elements has been studied experimentally in a number of papers.[1-5] It has been found that in the heavy-element region the observed yield of π^+ mesons increases with A more slowly that would follow from an $A^{2/3}$ law (whereas for the case of photoproduction of π-mesons the $A^{2/3}$ law is obeyed over a very wide range of variation of A).

As an explanation of this fact it is natural to suggest[4] that in addition to the large probability for absorption of a π meson produced inside the nucleus, which gives the $A^{2/3}$ law (only the surface nuclei play an effective part in the production), there is also an appreciable absorption of the protons penetrating into the nucleus. By absorption one obviously means here an effective attenuation of the beam of protons capable of producing π mesons. Some other possible causes of the deviation from proportionality to $A^{2/3}$ will be considered below

If we assume that for a given energy of the incident protons there are produced per unit volume of the nucleus, independently of its atomic number, the number $w(\theta)\sin\theta d\theta d\varphi$ of mesons emitted in the direction (θ, φ) relative to the incident beam, then the cross-section for π meson production from the nucleus can be written in the form

$$d\sigma/d\Omega = w(\theta)\int e^{-\varkappa s_1-\eta s_2}\,dV, \tag{1}$$

where s_1 and s_2 are the paths of the meson and proton inside the nucleus, and κ and η are their respective absorption coefficients. The integration is taken over the volume of the nucleus. This does not take account of effects of elastic scattering of meson and proton and of Coulomb interaction and the detailed momentum distribution of the nucleons in the nucleus.

In order of magnitude

$$w(\theta) \approx \sigma_0(\theta)\,A\,/\,{}^4/_3\pi R^3 = 3\sigma_0(\theta)\,/\,4\pi R_0^3, \tag{2}$$

where $\sigma_0(\theta)$ is the cross-section for single-nucleon production of π mesons and $R = R_0A^{1/3}$ is the radius of the nucleus.

In the absence of absorption of the proton ($\eta = 0$), Eq. (1) gives[6]

$$\frac{d\sigma}{d\Omega} = w(\theta)\,\frac{\pi R^2}{\varkappa}\left[1-\frac{1}{2(\varkappa R)^2}+\frac{e^{-2\varkappa R}}{2(\varkappa R)^2}(1+2\varkappa R)\right], \tag{3}$$

from which we have for $\kappa R \gg 1$: $d\sigma/d\Omega \sim R^2 \sim A^{2/3}$.

The validity of the $A^{2/3}$ law for small A means that we can regard the meson absorption as the larger through practically the entire periodic table. Under the condition $\kappa R \gg 1$ (for all values of R in question) and $\kappa \gg \eta$, Eq. (1) reduces to the form:

$$d\sigma\,/\,d\Omega = C(\theta)\,F(\eta R;\theta),\quad C(\theta) \equiv w(\theta)\,\pi\,/\,\varkappa\eta^2 \approx 3\sigma_0(\theta)\,/\,4\varkappa\eta^2R_0^3,\quad F(\eta R;\theta) = (\eta R)^2\sin^2\frac{\theta}{2}+\cos\theta\left[\frac{1}{4}-e^{-2\eta R}\left(\eta R+\frac{1}{2}\right)\right]$$
$$+\,I(\eta R;\theta)\quad\left(\theta\leqslant\frac{\pi}{2}\right),\quad F(\eta R;\theta) = (\eta R)^2\sin^2\frac{\theta}{2}+\frac{1}{4}\cos\theta+I(\eta R;\theta)\quad\left(\theta\geqslant\frac{\pi}{2}\right),$$

$$I(\eta R;\theta)=\frac{\cos^2\theta}{2\pi}\int_0^{\pi/2}\frac{\exp\{-2\eta R\sin\theta\sin\alpha\}}{1-\sin^2\theta\sin^2\alpha}\,d\alpha+\frac{\sin\theta\cos^2\theta}{\pi}(\eta R)\int_0^{\pi/2}\frac{\exp\{-2\eta R\sin\theta\sin\alpha\}\sin\alpha}{1-\sin^2\theta\sin^2\alpha}\,d\alpha$$
$$+\frac{2\sin^2\theta}{\pi}(\eta R)^2\int_0^{\pi/2}\exp\{-2\eta R\sin\theta\sin\alpha\}\cos^{-2}\alpha\,d\alpha. \tag{4}$$

The structure of the formula (4) is obvious. Under the condition $\kappa R \gg 1$ only a thin layer of thickness $\sim 1/\kappa$ at the surface of the nucleus makes any contribution to the cross-section. The part of the surface of the nucleus which is in front in relation to the incident protons and visible at the angle θ, on which the protons fall directly, is responsible for the term $(\eta R)^2 \sin^2\theta/2$ (this part of the surface of the sphere is contained between two planes passing through the center of the nucleus and perpendicular to the directions of the incident proton beam and the propagation of the mesons). The other terms are due to the production of π mesons near that part of the surface reached by a partly attenuated beam of protons.

For the angles $\theta = 0, \pi/2$, and π, Eq. (4) gives

$$F(\eta R,0)=\frac{1}{2}-e^{-2\eta R}\left(\eta R+\frac{1}{2}\right),\quad F\left(\eta R,\frac{\pi}{2}\right)=\frac{1}{2}(\eta R)^2+\frac{2}{\pi}(\eta R)^2\int_0^{\pi/2}e^{-2\eta R\sin\alpha}\cos^2\alpha\,d\alpha,\quad F(\eta R;\pi)=(\eta R)^2. \tag{5}$$

For $\eta R \lesssim 1$:

$$F(\eta R;\theta)=(\eta R)^2\Big\{1-(\eta R)\frac{4}{3\pi}[(\pi-\theta)\cos\theta+\sin\theta]+(\eta R)^2\cos^4\frac{\theta}{2}-(\eta R)^3\frac{8}{15\pi}\left[(\pi-\theta)\cos\theta+\sin\theta\left(1-\frac{\sin^2\theta}{3}\right)\right]$$
$$+(\eta R)^4\frac{2}{9}\cos^6\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2}\right)+\cdots\Big\}. \tag{6}$$

For $\eta R \sin\theta \gg 1$:

$$F(\eta R;\theta)=(\eta R)^2\sin^2\frac{\theta}{2}+\eta R\frac{\sin\theta}{\pi}+\frac{1}{4}\cos\theta+\cdots. \tag{7}$$

Figure 1 shows curves of the function $F(\eta R, \theta)$ plotted against $(\eta R)^2$ for $\theta = 0, \pi/4, \pi/2, 3\pi/4$, and π.

Before going on to the comparison of our results with the experimental data, we note one more circumstance.[4] If the cross-section for production of π^+ mesons on protons is much larger than that for the production on neutrons, then in the expression given in Eq. (4) for $C(\theta)$ we should insert the additional factor Z/A, where Z is the number of protons in the nucleus. At the beginning of the periodic table this factor is about 1/2 and is almost unchanged as A varies, but for the heavy elements Z can be considerably smaller than $A/2$. To take account of this effect there are plotted in Figs. 2–4, besides the experimental values of the cross-sections (circles), also the values of the cross-sections multiplied by A/Z (crosses). Good agreement with experiment can be regarded as existing when the theoretical curve passes somewhere between the circles and the crosses.

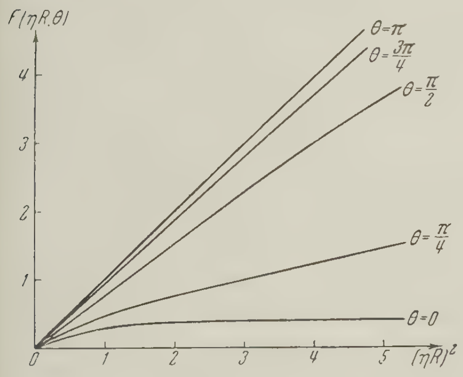

FIG. 1

Figure 2 shows the data from the work of Meshkovskii and others.[5] Ordinates are values of the total (with respect to the energies of the π^+ mesons produced) cross-section for π-meson production, measured in millibarns per steradian. The measurements were carried out at angle 45° with the direction of incidence of the proton beam, with initial proton energy 660 Mev. The errors for the values of the cross-sections multiplied by $A/2Z$ are not

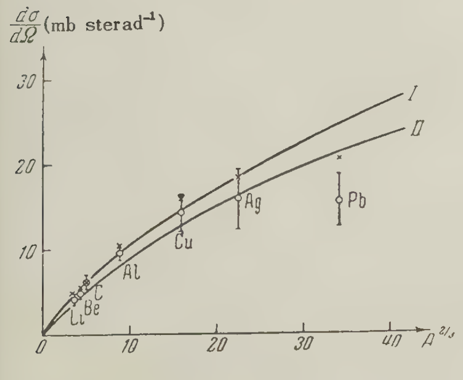

FIG. 2.

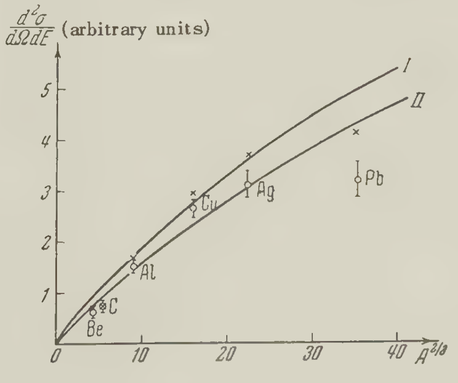

FIG. 3.

drawn in, for greater clearness in reading the diagram. Figure 3 shows the data of Sagane and others.[3] Ordinates are values of $d^2\sigma/d\Omega dE$ at $\theta = 90°$ and energy $E_{\pi+} = (33 \pm 3)$ Mev; the proton energy used in the experiment was $E_p = 340$ Mev. Figure 4 shows the data of Clark;[1] $\theta = 130 - 150°$, $E_{\pi+} = 40$ Mev, $E_p = 240$ Mev. The theoretical curve is drawn for the angle 135°. In all the diagrams the theoretical curve I corresponds to the value $\eta R_0 = 0.28$, and curve II to $\eta R_0 = 0.32$. The value of $C(\theta)$ in Eq. (4) was chosen so that curve I should go through the cross for Ag. As can be seen from the diagrams, within the limits of the present errors the theoretical results are in adequate agreement with experiment. The best agreement is given by curve I compared with the cross-sections multiplied by A/2Z. Comparisons with the data of Merritt and Hamlin[4] ($\theta = 0°$) and of Block and others[2] ($\theta = 90°$) were not carried out because of the small number of experimental points in the region of heavy nuclei. Some systematic reduction of the cross-section for Pb is possibly explained by an increased density of nuclear matter in lead (doubly magic nucleus).

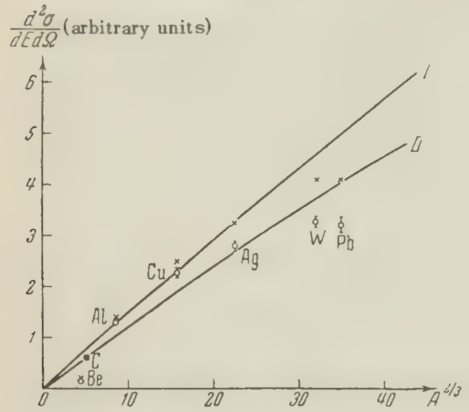

FIG. 4.

We note that according to these curves the deviation from the $A^{2/3}$ law exists already for the lightest elements, so that even if one draws a straight line through the first few points (Li — Al) it lies at an appreciable angle with the line that one would have in the absence of attenuation of the proton beam.

In conclusion we express our deep gratitude to K. A. Ter-Martirosian for proposing the problem and for discussions.

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CORRELATION IN SOLID SOLUTIONS

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A method is proposed for describing the non-equilibrium states of solid solutions which is based on the furnishing of a system of correlation functions for the dissolved atoms. A method of determining the free energy of solid solutions in states of total or partial equilibrium is also proposed. In the general case, the free energy is expressed in terms of a double, triple, etc., "interaction" of impurity atoms, which are assumed known. For isotopic mixtures, the "vibrational interaction" energy is determined in closed form and the entire problem can be carried through to its conclusion. The correlation functions obtained for the dissolved atoms are determined with the aid of the expressions thus constructed. The effect of correlation on the scattering of neutrons by isotope solutions is ascertained

INTRODUCTION

THE thermodynamic properties of solid solutions can be investigated by means of methods which are to a certain degree similar to the methods developed in the theory of non-ideal gases. In such a case, use is made of the concept of the "interaction" of impurity atoms. By the "interaction" of n atoms of an impurity we mean the difference between the free energy of a system with n impurity atoms, located at some distance from one another, and the same quantity for impurity atoms that are infinitely far apart. It is obvious that these interactions are rapidly decreasing functions of the distances between the atoms.

The determination of such a type of interaction is separate problem. In the case of solutions of atoms which differ markedly from one another, the principal part of the energy of interaction is its static part. (In this case, usually, only the interaction of nearest neighbors is taken into account.) If the potential energy of the interaction does not depend on the type of atoms (isotope solutions), the vibrational part of the interaction plays the chief role. Its determination can be brought about by means of the method developed earlier.[1]

Considering the interaction as given, we have succeeded in obtaining a general expression for the equilibrium free energy of the solution.

Prescription of the non-equilibrium states is obtained through the use of a system of correlation functions which determine the probabilities relative to the location of the dissolved atoms. To date, so far as we know, there exists no such method for the description of the non-equilibrium states. The method developed here permits us to compute the free energy of the solution in a state of partial equilibrium.

As an example, we have obtained an explicit expression for the non-equilibrium free energy for the case in which the lack of equilibrium arises as the result of the quenching of the solution from the temperature Θ' to the temperature Θ.

The correlation functions of the impurity atoms are obtained with the help of expressions computed for the free energy. For isotope solutions, the energy for the vibrational interaction of which is computed in closed form, the determination of the correlation can be carried through to its conclusion.

We begin with the calculation of the equilibrium values of the free energy of the solution.

1. EQUILIBRIUM FREE ENERGY

The equilibrium free energy is given, as is well known, by the expression

$$F - = -\Theta \ln Z; \quad Z = \sum_{\mathfrak{M}} \sum_{E_\alpha(\mathfrak{M})} \exp\left\{-\frac{E_\alpha(\mathfrak{M})}{\Theta}\right\} = \sum_{\mathfrak{M}} \exp\left\{-\frac{F(\mathfrak{M})}{\Theta}\right\}. \tag{1}$$

Here $\Theta = kT$, k = Boltzmann's constant, T = temperature $\mathfrak{M}$ = configuration of the impurity atoms, $E_\alpha(\mathfrak{M})$ is the energy level of the system for a given configuration, $F(\mathfrak{M})$ is the free energy of the solution for the given configuration. For n impurity atoms, the function $F(\mathfrak{M})$ can be expressed in the form

$$F(\mathfrak{M}) = F_0 + n\psi_1 + U_n(\mathfrak{M}), \quad U_n(\mathfrak{M}) = \psi_n(\mathfrak{M}) - n\psi_1. \tag{2}$$

F_0 is the free energy of the pure solvent ψ_1 is the change in the free energy upon the substitution of a single atom of impurity for an atom of solvent, $\psi_n(\mathfrak{M})$ is the change in the free energy upon replacing n atoms, $U_n(\mathfrak{M}) = U_n(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ is the free energy of interaction of n impurity atoms located at the sites $\mathbf{r}_1, \ldots, \mathbf{r}_n$. In the case of a simple lattice, where the interaction of the particles depends only on the mutual distance between them, U_n is completely determined by the position of $n-1$ particles relative to a single individual [$U_2 = U_2(\mathbf{r}_2 - \mathbf{r}_1)$, $U_3 = U_3(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{r}_3 - \mathbf{r}_1)$ etc.]. It is evident that only the last term depends on the configuration in (2). According to (1) and (2), we have for the free energy of the solution in the case of dissolved atoms:

$$F = F_0 + n\psi_1 + N\Theta[c\ln c + (1-c)\ln(1-c)] - N\Theta\Phi_n, \quad \Phi_n = \Phi(c) = \frac{1}{N}\ln\overline{e^{-U_n/\Theta}}, \tag{3}$$

$$\overline{e^{-U_n/\Theta}} = 1 + \Delta_n, \quad \Delta_n = \frac{1}{\mathfrak{N}}\sum_{\mathfrak{M}}\left(\exp\left\{-\frac{U_n(\mathfrak{M})}{\Theta}\right\} - 1\right). \tag{4}$$

Here N is the total number of particles in the system, $c = n/N$, $\mathfrak{N} = N!/(N-n)!\,n!$ is the total number of configurations: Φ_n, as a function of the concentration c, does not have any singularities and can be represented in the form of a series in powers of c. In this case, we make use of the asymptotic relation (valid for $N \to \infty$)

$$\Phi_n = e^{c\hat{Q}}\,\Phi_k|_{k=0}, \tag{5}$$

where $\hat{Q}$ is the difference operator, analogous to the operator of differentiation with respect to the concentration. If $\hat{T}$ is the operator which changes the number of impurity atoms by unity, then

$$\hat{Q} = N(\hat{T} - 1). \tag{6}$$

(The change in the concentration upon the addition of one particle is equal to 1/N.) Equation (5) represents the discrete analogue of the formula for a function of continuous parameters

$$f(x+c) = e^{c\partial/\partial x} f(x)$$

and is obtained from it by replacing the operator $\partial/\partial x$ by the operator $\hat{Q}$. With accuracy up to terms proportional to c^3, the expression takes the form

$$\Phi_n = \left[N^2\frac{c^2}{2}\hat{T}^2 + N^3\frac{c^3}{3!}(\hat{T}^3 - 3\hat{T}^2)\right]\Phi_k|_{k=0}. \tag{7}$$

(Application of the unity operator and the operator $\hat{T}$ to $\Phi_k|_{k=0}$ yields zero, since $\Phi_0 = \Phi_1 = 0$.) Thus, we get*

$$\Phi_n = N^2\frac{c^2}{2}\ln(1+\Delta_2) + N^3\frac{c^3}{3!}[\ln(1+\Delta_3) - 3\ln(1+\Delta_2)], \tag{8}$$

$$\Delta_2 = \frac{1}{N}\sum_{\mathbf{r}}\left(\exp\left\{-\frac{U_2(\mathbf{r})}{\Theta}\right\} - 1\right), \quad \Delta_3 = \frac{1}{N(N-1)}\sum_{\mathbf{r},\mathbf{r}'}\left(\exp\left\{-\frac{U_3(\mathbf{r},\mathbf{r}')}{\Theta}\right\} - 1\right). \tag{9}$$

Here and below, the terms with $\mathbf{r} = 0$, $\mathbf{r}' = 0$, $\mathbf{r} = \mathbf{r}'$, etc. are excluded from the summation over $\mathbf{r}$ and $\mathbf{r}'$. Inasmuch as Δ_2 and Δ_3 are quantities of the order of 1/N, we can carry out a logarithmic expansion in (8). We note that in each successive term of (8) the expansion ought to be carried out with greater accuracy. For example, in the component $\sim c^2$ in the logarithmic expansion, it suffices to limit ourselves to the term of first order of smallness; in the component $\sim c^3$, the expansion ought to be carried out up to terms of order $1/N^2$. This circumstance is determined by the fact that in the expression

$$\ln(1+\Delta_3) - 3\ln(1+\Delta_2)$$

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*For simplicity here and in what follows (with the exception of Sec. 3) it is assumed that the lattice is simple. Generalization to the case of a complicated lattice presents no special difficulties.

the terms of first order of smallness vanish. Carrying through the expansion, we get

$$\Phi_n = \Phi(c) = \frac{c^2}{2} f_2 + \frac{c^3}{3!}(f_3 - 3f_2^2), \tag{10}$$

where

$$f_2 = \sum_{\mathbf{r}} \left(\exp\left\{-\frac{U_2(\mathbf{r})}{\Theta}\right\} - 1\right), \quad f_3 = \sum_{\mathbf{r},\mathbf{r}'} \left(\exp\left\{-\frac{U_3(\mathbf{r},\mathbf{r}')}{\Theta}\right\} - 3\exp\left\{-\frac{U_2(\mathbf{r})}{\Theta}\right\} + 2\right). \tag{11}$$

The equilibrium free energy of the solution is equal to

$$F = F_0 + Nc\psi_1 + N\Theta\,[c \ln c + (1-c)\ln(1-c)] - N\Theta\left[\frac{c^2}{2} f_2 + \frac{c^3}{3!}(f_3 - 3f_2^2)\right]. \tag{12}$$

with the same accuracy. Thus the free energy of a solution is a functional of the interaction of two, three, etc., impurity atoms. Here the terms proportional to c^k are determined only by interactions of no more than k particles.

We can also represent the free energy in the form of a functional of pair interactions φ_2 and "polarization corrections" of third, fourth, etc., orders φ_k. It suffices here to make use of the relation

$$U_n(\mathbf{r}_1, \ldots, \mathbf{r}\ ) = \frac{1}{2}\sum_{i \neq k} \varphi_2(\mathbf{r}_i - \mathbf{r}_k) + \frac{1}{3!}\sum_{i \neq k \neq l} \varphi_3(\mathbf{r}_i - \mathbf{r}_l, \mathbf{r}_k - \mathbf{r}_l) + \ldots + \varphi_n(\mathbf{r}_2 - \mathbf{r}_1, \ldots, \mathbf{r}_n - \mathbf{r}_1). \tag{13}$$

Each of the $\mathbf{r}$ in the sums here run over all values of the radius vectors of the impurity atoms $(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ which are present for the given configuration. In particular, we have the relation

$$U_2(\mathbf{r}) = \varphi_2(\mathbf{r}),\ U_3(\mathbf{r}, \mathbf{r}') = \varphi_2(\mathbf{r}) + \varphi_2(\mathbf{r}') + \varphi_2(\mathbf{r} - \mathbf{r}') + \varphi_3(\mathbf{r}, \mathbf{r}'),$$

and conversely,

$$\varphi_2(\mathbf{r}) = U_2(\mathbf{r}),\ \varphi_3(\mathbf{r}, \mathbf{r}') = U_3(\mathbf{r}, \mathbf{r}') - U_2(\mathbf{r}) - U_2(\mathbf{r}') - U_2(\mathbf{r} - \mathbf{r}').$$

From the expression we have obtained for the free energy of solution, it is easy to determine the chemical potentials μ of the solvent and μ' of the dissolved material:

$$\begin{aligned} \mu &= \mu_0 + \Theta \ln(1-c) + \Theta\left[\frac{c^2}{2} f_2 + \frac{c^3}{3}(f_3 - 3f_2^2)\right], \\ \mu' &= \mu_0 + \psi_1 + \Theta \ln c - \Theta\left[c f_2 + \frac{c^2}{2}(f_3 - 3f_2^2 - f_2) - \frac{c^3}{3}(f_3 - 3f_2^2)\right]. \end{aligned} \tag{14}$$

Here μ_0 is the chemical potential of the pure solvent.

2. NON-EQUILIBRIUM STATES OF SOLID SOLUTIONS. CALCULATION OF THE FREE ENERGY, ENTROPY, AND CORRELATION FUNCTIONS

Keeping in mind the determination of the correlation functions, we compute the free energy of the solution in a state of particle equilibrium. The computation of such a quantity is of independent interest, inasmuch as the configuration of solid solutions is usually not an equilibrium one (because of the slowness of processes of rearrangement of the atoms).

First, it is necessary to establish the method of description of the macroscopic non-equilibrium states of solid solutions. As is shown, the most complete description of such a state can be obtained by giving the collection of correlation functions W_k connected with the probabilities $P(\mathbf{r}_1, \ldots, \mathbf{r}_k)$ of locating k foreign atoms at the sites $\mathbf{r}_1, \ldots, \mathbf{r}_k$ by the relations

$$P(\mathbf{r}_1, \ldots, \mathbf{r}_k) = c^k W_k(\mathbf{r}_2 - \mathbf{r}_1, \ldots, \mathbf{r}_k - \mathbf{r}_1). \tag{15}$$

In practice, it suffices to give the first few W_k (i.e., $k = 2, 3, 4$). This follows from the fact that the functions $\varphi_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1})$ in (18) rapidly fall off with the distances $\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}$ as well as with the number k.

The free energy of a non-equilibrium solution is determined in this case by the expression

$$F\{W\} = -\Theta \sum_{\mathfrak{M}\{W\}} \exp\left\{-\frac{F(\mathfrak{M})}{\Theta}\right\}, \tag{16}$$

where the summation runs only over the configurations which correspond to the given values of W. If $M_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1})$ denotes the number of groups of k impurity atoms with distances $\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}$ between them, then, by Eq. (15), we have

$$M_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}) = Nc^k W_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}). \tag{17}$$

On the other hand, for an arbitrary configuration, the free energy is equal to

$$\begin{aligned} F(\mathfrak{M}) &= F_0 + Nc\psi_1 + \sum_k \frac{1}{k!} \sum_{\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}} M_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1})\, \varphi_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}) \\ &= F_0 + Nc\psi_1 + N \sum_k \frac{c^k}{k!} \sum_{\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}} W_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1})\, \varphi_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}). \end{aligned} \tag{18}$$

Here $\varphi_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1})$ is the pair interaction $(k = 2)$ and the polarization correction $(k = 2)$ determined above. This signifies that the $F(\mathfrak{M})$ are the same for all configurations corresponding to the same values of W. Therefore, we have

$$F(W) = F_0 + Nc\psi_1 + N \sum_k \frac{c^k}{k!} \sum_{\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}} W_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1})\, \varphi_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}) - \Theta\sigma\{W\}. \tag{19}$$

Here $\sigma\{W\} = -\ln \mathfrak{N}\{W\}$ is the configurational part of the entropy as a functional of the correlation functions, and $\mathfrak{N}\{W\}$ is the number of admissible configurations.

For equilibrium values of the correlation functions $\widetilde{W}_k$, the free energy acquires the value

$$F = F_0 + Nc\psi_1 + N \sum_k \frac{c^k}{k!} \sum_{\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}} \widetilde{W}_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1})\, \varphi_k(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}) - \Theta\sigma\{\widetilde{W}\}, \tag{20}$$

which, on the other hand, is determined by Eqs. (11) and (12). Inasmuch as the free energy is a minimum for the equilibrium values of $\widetilde{W}$, the relation

$$N\frac{c^k}{k!}\varphi_k - \Theta\delta\sigma\{\widetilde{W}\}/\delta\widetilde{W}_k = 0, \tag{21}$$

holds, where the functional derivative appears in the second term. Taking (21) into account, it is easy to see that $\widetilde{W}_k$ and $\sigma\{\widetilde{W}\}$ are determined from the equilibrium energy by means of the relations

$$N\frac{c^k}{k!}\widetilde{W}_k = \delta F/\delta\varphi_k\big|_{\Theta}, \tag{22}$$

$$\sigma\{W\} = -\partial F/\partial\Theta\big|_{F_0,\, \psi_1,\, \varphi_k}. \tag{23}$$

Equations (22) and (23) determine the equilibrium correlations and the configurational part of the entropy of solution in terms of the function φ_k.

If we use Eq. (12) for F, then we get

$$\widetilde{W}_2(\mathbf{r}) = e^{-\varphi_2(\mathbf{r})/\Theta} + c\sum_{\mathbf{r}'}\left(e^{-U_3(\mathbf{r}, \mathbf{r}')/\Theta} - 2e^{-[\varphi_2(\mathbf{r}) + \varphi_2(\mathbf{r}')]/\Theta} + e^{-\varphi_2(\mathbf{r})/\Theta}\right). \tag{22a}$$

$$\widetilde{W}_3(\mathbf{r}, \mathbf{r}') = e^{-U_3(\mathbf{r}, \mathbf{r}')/\Theta} \quad (U_3(\mathbf{r}, \mathbf{r}') = \varphi_2(\mathbf{r}) + \varphi_2(\mathbf{r}') + \varphi_2(\mathbf{r} - \mathbf{r}') + \varphi_3(\mathbf{r}, \mathbf{r}')),$$

$$\begin{aligned} \sigma = -N\Big\{&c\ln c + (1-c)\ln(1-c) + \frac{c^2}{2}\sum_{\mathbf{r}}\left[1 - e^{-\varphi_2(\mathbf{r})/\Theta}\left(1 + \frac{\varphi_2(\mathbf{r})}{\Theta}\right)\right] \\ &+ \frac{c^3}{3!}\sum_{\mathbf{r}, \mathbf{r}'}\left[1 - e^{-U_3(\mathbf{r}, \mathbf{r}')/\Theta}\left(1 + \frac{U_3(\mathbf{r}, \mathbf{r}')}{\Theta}\right) - 3e^{-\varphi_2(\mathbf{r})/\Theta}\left(1 + \frac{\varphi_2(\mathbf{r})}{\Theta}\right) + 3e^{-[\varphi_2(\mathbf{r}) + \varphi_2(\mathbf{r}')]/\Theta}\left(1 + \frac{2\varphi_2(\mathbf{r})}{\Theta}\right)\right] + \ldots\Big\}. \end{aligned} \tag{23a}$$

We can consider Eqs. (22a) and (23a) as parametric expressions for the entropy. The expression for the entropy in terms of the correlation functions is obtained by solving (22a) for φ_2 and φ_3, and substituting the resultant values in (23a):

$$\varphi_2(\mathbf{r}) = -\Theta\left\{\ln\widetilde{W}_2(\mathbf{r}) - \frac{c}{\widetilde{W}_2(\mathbf{r})}\sum_{\mathbf{r}'}[\widetilde{W}_3(\mathbf{r}, \mathbf{r}') - 2\widetilde{W}_2(\mathbf{r})\widetilde{W}_2(\mathbf{r}') + \widetilde{W}_2(\mathbf{r})],\right. \quad U_3(\mathbf{r}, \mathbf{r}') = -\Theta\ln\widetilde{W}_3(\mathbf{r}, \mathbf{r}'),$$

$$\sigma\{W\} = -N\left\{c\ln c + (1-c)\ln(1-c) + \frac{c^2}{2}\sum_{\mathbf{r}}[W_2(\mathbf{r})\ln W_2(\mathbf{r}) - W_2(\mathbf{r}) + 1] + \right.$$

$$+\frac{c^3}{3!}\sum_{\mathbf{r},\mathbf{r}'}[W_3(\mathbf{r},\mathbf{r}')\ln W_3(\mathbf{r},\mathbf{r}')-W_3(\mathbf{r},\mathbf{r}')+1-3(W_3(\mathbf{r},\mathbf{r}')\ln W_2(\mathbf{r})-W_2(\mathbf{r})W_2(\mathbf{r}')+W_2(\mathbf{r}'))]+\ldots\Big\}. \tag{23b}$$

[The sign of the equilibrium of the correlation functions is omitted in (23b), inasmuch as the resultant expression for the entropy is valid (as is easy to show) for arbitrary values of W_k.]

In the case in which all the polarization corrections are equal to zero (the interaction of the particles is composed of their pair interactions) it suffices, for the determination of the non-equilibrium state, to give only the binary correlation functions W_2. The remaining functions are determined from the condition of a minimum for the free energy (21) and is given in terms of W_2.

In particular, for W_3, Eq. (21) yields in this case

$$W_3(\mathbf{r},\ \mathbf{r}')=W_2(\mathbf{r})W_2(\mathbf{r}')W_2(\mathbf{r}-\mathbf{r}').$$

[In using (21), it should be recalled that the expression under the sign of the double sum in (23b) should be symmetrized relative to a permutation of the particles.] The expression for the entropy in this case is materially simplified and has the form

$$\sigma=-N\Big\{c\ln c+(1-c)\ln(1-c)+\frac{c^2}{2}\sum_{\mathbf{r}}[(1+V_2(\mathbf{r}))\ln(1+V_2(\mathbf{r}))-V_2(\mathbf{r})]-\frac{c^3}{3!}\sum_{\mathbf{r},\mathbf{r}'}V_2(\mathbf{r})V_2(\mathbf{r}')V_2(\mathbf{r}-\mathbf{r}')+\cdots\Big\}. \tag{24}$$

Here

$$V_2(\mathbf{r})=W_2(\mathbf{r})-1.$$

We note that in this case the entropy is specified parametrically by means of only two relations. Actually, Eq. (23) determines the entropy as a functional only of the pair interaction $\varphi_2(\mathbf{r})$, which can be determined from the expression for the binary correlation function $W_2(\mathbf{r})$.

The free energy of a solid solution corresponding to a non-equilibrium configuration is determined from Eq. (19) by substitution in it of the expression for the entropy, found with the necessary accuracy.

Let us consider the special case in which the non-equilibrium comes about as the result of a quenching of the solution from the temperature Θ' to the temperature Θ. Obviously the configuration of the solution is not changed in the quenching, and the values of the correlation coefficients in the non-equilibrium state thus obtained are equal to their equilibrium values at the temperature Θ'. The free energy in this case is equal to

$$F(\Theta,\Theta',c)=F_0(\Theta)+Nc\psi_1(\Theta)+N\sum_k\frac{c^k}{k!}\sum_{\mathbf{r}_1,\ldots,\mathbf{r}_{k-1}}\widetilde{W}_k(\mathbf{r}_1,\ldots,\mathbf{r}_{k-1},\Theta')\varphi_k(\mathbf{r}_1,\ldots,\mathbf{r}_{k-1},\Theta)-\Theta\sigma\{\widetilde{W}_k\}(\Theta')\Big\}. \tag{25}$$

If we substitute the explicit expressions for the correlation functions and the entropy, we then obtain (with accuracy up to terms $\sim c^2$)

$$F(\Theta,\Theta',c)=F(\Theta,c)+N\frac{c^2}{2}\sum_{\mathbf{r}}\Big\{e^{-\varphi_2(\mathbf{r},\Theta')/\Theta'}\Big[\varphi_2(\mathbf{r},\Theta)-\frac{\Theta}{\Theta'}\varphi_2(\mathbf{r},\Theta')\Big]-\Theta\,[e^{-\varphi_2(\mathbf{r},\Theta')/\Theta'}-e^{-\varphi_2(\mathbf{r},\Theta)/\Theta}]\Big\}.$$

Here $F(\Theta, c)$ is the equilibrium free energy of the solution.

3. SOLUTIONS OF ISOTOPES

The method just developed for the determination of the free energy is applicable for solutions of general form. In this case the interactions of the impurity atoms U_k or, what amounts to the same thing, the φ_k, ought to be given, and are the parameters of the system. Here we have a case in which all the calculations can be carried through to completion. The latter applies to a solution of isotopes, for which the functions U_n (and the φ_n) are the energy of vibrational interaction of the impurity isotope and can be expressed in terms of the spectrum of frequencies of an ideal crystal (i.e., the lattices of the pure isotope).*

The change in the vibrational part of the free energy in the case in which there are n atoms of the principal isotope with mass m, located at the lattice sites $\mathbf{r}_1, \ldots, \mathbf{r}_n$, are replaced by impurity atoms with mass m', is given by the expression[1]

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*Part of the results contained in Ref. 1 was later obtained independently by Montroll and Potts[2] who determined the energy of the vibrational interaction of the defects of the lattice.

$$\psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \int \xi_n(z, \mathbf{r}_1, \ldots, \mathbf{r}_n)\, \chi'(z)\, dz, \tag{26}$$

$$\xi_n = \frac{1}{\pi} \arg D\left(\delta_{pq}\delta_{xx'} + \varepsilon z \int I^{xx'}_{\mathbf{r}_p - \mathbf{r}_q}(\mu) \frac{d\mu}{\mu - z - i0}\right) (p, q = 1, 2, \ldots, n;\ x, x' = 1, 2, 3), \tag{27}$$

$$I^{xx'}_{\mathbf{r}}(\mu) = \sum_j \int\limits_{\omega_j^2(\mathbf{k}) = \mu} \frac{d\Omega_j}{|\nabla \omega_j^2|} e^{2\pi i \mathbf{k}\mathbf{r}} u_j^x(\mathbf{k})\, u_j^{x'*}(\mathbf{k}'),\quad \varepsilon = \frac{m - m'}{m},\quad \chi(z) = \Theta \ln\left(1 - e^{-\hbar\sqrt{z}/\Theta}\right) + \frac{1}{2}\hbar\sqrt{z}. \tag{28}$$

Here $\mathbf{k}$ is the wave vector, $\omega_j(\mathbf{k})$ is the j-th member of the frequencies of the pure fundamental isotope, $\mathbf{u}_j(u_j^1, u_j^2, u_j^3)$ is the normalized polarization vector $d\Omega_j$ is the element of area of the surface $\omega_j^2(\mathbf{k}) = \mu$ over which the integration is carried out.* The interaction of n impurity atoms, as is easily seen, is equal to

$$U_n = \psi_n - n\psi_1 = \int (\xi_n - n\xi_1)\, \chi'(z)\, dz. \tag{29}$$

We find the expression for φ_k by means of Eq. (13). In particular, we have

$$\varphi_2(\mathbf{r}) = \psi_2(\mathbf{r}) - 2\psi_1,\quad \varphi_3(\mathbf{r}, \mathbf{r}') = \psi_3(\mathbf{r}, \mathbf{r}') - \psi_2(\mathbf{r}) - \psi_2(\mathbf{r}') - \psi_2(\mathbf{r} - \mathbf{r}') + 3\psi_1.$$

For small values of ϵ, these relations become materially simplified. With an accuracy up to terms $\sim \epsilon^2$ we have, for z lying in the region of the continuous spectrum of the pure solvent,

$$\begin{aligned} \xi_n(z, \mathbf{r}_1, \ldots, \mathbf{r}_n) = n\left[\varepsilon z \sum_x I_0^{xx}(z) - \varepsilon^2 z^2 \operatorname{Re} \sum_{x, x'} I_0^{xx'}(z) \int' I_0^{x'x}(\mu) \frac{d\mu}{\mu - z}\right] \\ -\varepsilon^2 z^2 \operatorname{Re} \sum_{p \neq q} \sum_{x, x'} I^{xx'}_{\mathbf{r}_p - \mathbf{r}_q}(z) \int' I^{x'x}_{\mathbf{r}_q - \mathbf{r}_p}(\mu) \frac{d\mu}{\mu - z}\quad (p, q = 1, 2, \ldots, n;\ x, x' = 1, 2, 3), \end{aligned} \tag{30}$$

Here $\int'$ is the principal value of the integral. For the given values of z,

$$\int I^{xx'}_{\mathbf{r}}(\mu) \frac{d\mu}{\mu - z - i0} = \int' I^{xx'}_{\mathbf{r}}(\mu) \frac{d\mu}{\mu - z} + i\pi I^{xx'}_{\mathbf{r}}(z).$$

Outside the region of the continuous spectrum, $\xi_n = 0$ in the case of sufficiently small ϵ. It is easy to be convinced that (30) can be written in the form

$$\xi_n(z, \mathbf{r}_1, \ldots, \mathbf{r}_n) = n\xi_1(z) + \frac{1}{2} \sum_{p \neq q} (\xi_2(z, \mathbf{r}_p - \mathbf{r}_q) - 2\xi_1(z)\ (p, q = 1, 2, \ldots, n). \tag{31}$$

Then, taking Eq. (29) into account, we obtain

$$U_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \frac{1}{2} \sum_{p \neq q} U_2(\mathbf{r}_p - \mathbf{r}_q). \tag{32}$$

Thus the interaction of an arbitrary number of atoms of the impurity isotope, with accuracy up to terms $\sim \epsilon^2$, is composed of their pair interactions.† The interaction of two impurity atoms is equal (with the same accuracy) to

$$U_2(\mathbf{r}) = -2\varepsilon^2 \operatorname{Re} \int dz \left[z^2 \chi'(z) \sum_{x, x'} I^{xx'}_{\mathbf{r}}(z) \int' I^{x'x}_{-\mathbf{r}}(\mu) \frac{d\mu}{\mu - z}\right. . \tag{33}$$

If we substitute the expression $\chi = h\sqrt{z}/2$ (which is valid for low temperatures) in Eq. (33) and change over to integration over the wave vector $\mathbf{k}$, we get

$$U_2(\mathbf{r}) = \frac{1}{4} \varepsilon^2 \hbar \operatorname{Re} \sum_{j, j'} \int d\mathbf{k}\, d\mathbf{k}' \frac{\omega_j^2(\mathbf{k}) + \omega_j(\mathbf{k})\, \omega_{j'}(\mathbf{k}') + \omega_{j'}^2(\mathbf{k})}{\omega_j(\mathbf{k}) + \omega_{j'}(\mathbf{k}')} e^{2\pi i (\mathbf{k} - \mathbf{k}')\mathbf{r}} |\, \mathbf{u}_j(\mathbf{k}),\ \mathbf{u}_{j'}(\mathbf{k}'))\, |^2. \tag{34}$$

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*In Ref. 1 the corresponding formulas were written for the idealized case of a simple lattice with a single direction of polarization, when only one frequency mode exists and the vector $\mathbf{u}$ is a unit vector in the direction of polarization. Formulas (42) and (43) correspond to the case of three directions of polarization in Ref. 1 for small ϵ.

† Similarly, it can be shown that with accuracy up to terms $\sim \epsilon^k$, the interaction of an arbitrary number of impurity atoms consists of their pair interactions and polarization corrections of order no higher than k.

To facilitate the estimate of the energy of interaction, we write its value in the idealized case of a simple lattice with a single direction of polarization:

$$U_2(\mathbf{r}) = -\varepsilon^2 \frac{\hbar}{4} \operatorname{Re} \int dk\, dk' \, \frac{\omega(\mathbf{k})\,\omega(\mathbf{k}')}{\omega(\mathbf{k}) + \omega(\mathbf{k}')} e^{2\pi i (\mathbf{k}-\mathbf{k}')\mathbf{r}}.$$

Substitution of the expressions obtained here for the interactions into the general formulas yields the thermodynamic functions and the equilibrium value of the correlation for solutions of isotopes.

In the general case of a complicated lattice, the expressions for the interactions of the impurity isotopes are somewhat more involved. Let the position of the p-th atom of the impurity be given by the radius vector $\mathbf{r}_p + \mathbf{r}(s_p)$, where $\mathbf{r}_p$ is determined by the fundamental cell of the lattice, while $\mathbf{r}(s_p)$ is the position of the impurity atom inside this cell. Then a change in the vibrational free energy upon the insertion of n atoms of impurity is determined by the expressions

$$\psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_n, s_1, \ldots, s_n) = \widetilde{\int} \xi_n(z, \mathbf{r}_1, \ldots, \mathbf{r}_n, s_1, \ldots, s_n)\,\chi'(z)\,dz; \tag{35}$$

$$\xi_n = \frac{1}{\pi} \arg D\left(\delta_{pq}\delta_{xx'} + \varepsilon z \int I_{\mathbf{r}_p-\mathbf{r}_q}^{s_p x,\, s_q x'}(\mu) \frac{d\mu}{\mu - z - i0}\right), \quad I_{\mathbf{r}}^{sx,\, s'x'}(\mu) = \sum_j \int_{\omega_j^2(\mathbf{k})=\mu} \frac{d\Omega_j}{|\nabla \cdot \omega_j^2|} e^{2\pi i \mathbf{k}\mathbf{r}} v_j^{sx}(\mathbf{k})\, v_j^{s'x'}(\mathbf{k}),$$
$$v_j^{sx} = u_j^{sx}(\mathbf{k})\, e^{2\pi i \mathbf{k}\mathbf{r}(s)}, \tag{36}$$

where ϵ and $\chi(z)$ are determined by Eqs. (27) and (28). In the case under consideration, ψ_1 depends on which atom of the basic cell is replaced by the impurity. Therefore, the interaction of n impurity atoms is determined by the relation

$$U_n = \psi_n - \sum_s n_s \psi_1(s),$$

where n_s is the number of impurity particles which replace atoms of type s of the basic lattice. It is easy to show that, with accuracy up to terms $\sim \epsilon^2$ the following relations hold

$$U_n(\mathbf{r}_1, \ldots, \mathbf{r}_n, s_1, \ldots, s_n) = \frac{1}{2} \sum_{p \neq q} U_2(\mathbf{r}_p - \mathbf{r}_q, s_p, s_q) \tag{37}$$

$$U_2(\mathbf{r}, s, s') = -2\varepsilon^2 \operatorname{Re} \int dz \left[z^2 \chi'(z) \sum_{x,\, x'} I_{-\mathbf{r}}^{sx,\, s'x'}(z) \int' I_{-\mathbf{r}}^{s'x',\, s'x'}(\mu) \frac{d\mu}{\mu - z} \right]. \tag{38}$$

Transformation to integration over the wave vector puts Eq. (36) in the following form for the low temperature case:

$$U_2(\mathbf{r}, s, s') = \varepsilon^2 \frac{\hbar}{4} \operatorname{Re} \sum_{j,\, j'} \int d\mathbf{k}\, d\mathbf{k}' \, \frac{\omega_j^2(\mathbf{k}) + \omega_j(\mathbf{k})\,\omega_{j'}(\mathbf{k}') + \omega_{j'}^2(\mathbf{k}')}{\omega_j(\mathbf{k}) + \omega_{j'}(\mathbf{k}')} e^{2\pi i (\mathbf{k}-\mathbf{k})\mathbf{r}} \left| (\mathbf{v}_j^s(\mathbf{k}),\ \mathbf{v}_{j'}^{s'}(\mathbf{k}')) \right|^2. \tag{39}$$

As must follow from the relations obtained, the interactions of the isotopes are very small. For the minimum separation of the atoms, in accordance with (33), the interaction has the value $\sim \epsilon^2 \Theta_D$ ($\Theta_D = kT_D$, T_D is the Debye temperature). Obviously, an appreciable correlation in the positions of the atoms will take place only at temperatures of the order or smaller than $\epsilon^2 T_D$. However, in a solid because of the slowness of the diffusion of the atoms at arbitrarily low temperatures, configurations actually exist which are in equilibrium at the temperature of solidification. Therefore, it is essential for the appearance of correlation in solutions of isotopes that the solidification itself take place at a temperature $T \sim \epsilon^2 T_D$. Hydrogen, for which $T_D \sim 100°$, $T_{so} \sim 20°$, $\epsilon \sim 1/2$ satisfies this condition. For other substances, sufficiently low temperatures of solidification can exist in the case of their solidification from the gaseous phase (in the sublimation region).

It should be pointed out that the lattice of hydrogen is molecular and the important role at energies $\varphi_2, \ldots$ is played not by vibrational but by rotational interactions, in which different statistics are essential. It should further be noted that in the general case still another circumstance can make a contribution to the energy of interaction of the isotopes. Because of the presence of anharmonics, the mean volume occupied by the atom (under consideration of the zero vibrations) depends on the amplitude, i.e., on the mass of the isotope. Therefore, upon substitution of an impurity atom at any lattice point, there arises an additional elastic energy. A semi-empirical estimate of this energy carried out for hydrogen by Prigogine et al.,[3] gives the value $U_2 \sim 10^{-15}$ erg, which corresponds to a temperature of $T \sim 10°$.

The presence of correlation measures the non-ideality of the solutions of isotopes. In particular, for sufficiently large values of the interaction, a separation of the solution takes place into phases; this was recently observed for impurities of hydrogen isotopes.*[4]

In the case in which the interaction is insufficient for a separation of the solution, the presence of correlation can be found with the aid of a study of slow neutron scattering. As is known, the probability that in passage through the crystal the neutron possesses the momentum $\mathbf{p}'$ if its initial momentum was $\mathbf{p}$ is proportional to the quantity

$$\overline{\left|\sum_{\mathbf{r}} A_{\mathbf{r}} \exp\left\{\frac{2\pi i}{\hbar}(\mathbf{p}'-\mathbf{p})\mathbf{r}\right\}\right|^2} = \overline{\sum_{\mathbf{r},\mathbf{r}'} (A_{\mathbf{r}}-\bar{A})(A_{\mathbf{r}'}-\bar{A})\exp\left\{\frac{2\pi i}{\hbar}(\mathbf{p}'-\mathbf{p})(\mathbf{r}-\mathbf{r}')\right\}} + \bar{A}^2\left|\sum_{\mathbf{r}}\exp\left\{\frac{2\pi i}{\hbar}(\mathbf{p}'-\mathbf{p})\mathbf{r}\right\}\right|^2. \quad (40)$$

Here $\mathbf{r}$ denotes the equilibrium position of the atoms of the crystal, $A_{\mathbf{r}}$ characterizes the interaction between neutrons and the nucleus at site $\mathbf{r}$ (in the case under analysis, A depends only on the mass). Averaging is carried out over 11 possible configurations. The second component in (40) determines the coherent scattering of the neutron. It gives sharp maximum of intensity of scattered neutrons, determined by the relation $\mathbf{p}' - \mathbf{p} = \hbar\tau$, here τ is the vector of the reciprocal lattice. The first term in (40) in the case of absence of correlation, when all terms with $\mathbf{r} \neq \mathbf{r}'$ reduce to zero, is equal to

$$N(\overline{A^2} - \bar{A}^2)$$

and determines the background noise which does not depend on the scattering angle. If correlation exists, the noise is equal to

$$N(\overline{A^2} - \bar{A}^2) + Nc_2(A_1 - A_2)^2 \sum_{\mathbf{r}} (W_2(\mathbf{r}) - 1)\exp\left\{\frac{2\pi i}{\hbar}(\mathbf{p}'-\mathbf{p})\mathbf{r}\right\}. \quad (41)$$

Here the indices 1 and 2 refer to isotopes of the solvent and impurity, respectively. According to (41) the noise is inhomogeneous in the presence of correlation, and we can determine the value of the correlation functions from the structure of the noise.

It should be observed that Eqs. (40), (41) are approximate, inasmuch as they were obtained without account of the thermal vibrations of the crystal. Account of the latter obviously does not reduce to the addition of a thermal factor, since the amplitudes of the vibration depend on the kind of isotope and on the correlation of the atoms of the lattice.

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*This conclusion (which did not appear in the published paper of Ref. 4, and which was kindly communicated to us by the authors) can be made on the basis of an analysis of x-rays of the mixture $H_2 - D_2$ with high concentration of both isotopes, in this case, two systems of lines are quite clearly seen.

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THEORY OF A DEGENERATE ELECTRON LIQUID

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Landau's theory of a Fermi liquid[1] is extended to the case of an electron liquid.

1. INTRODUCTION

THE electron theory of metals successfully uses for the description of many metallic properties the representation of the electrons as independent particles, and thus considers the electrons of the metal as a gas. True, since the electrons are located in the field of the metal lattice, their properties are considerably different from those of free electrons. This difference, in particular, is characterized by the dispersion law, i.e., by the dependence of the electron energy on their quasi-momentum. The difference in the energy of the electrons in the metal from the energy of free electrons is also due to the interaction between particles. This interaction is not at all small. Actually, the mean energy of Coulomb interaction of metal electrons is on the order of their mean kinetic energy. Therefore, it is natural to expect a substantial correlation in the motion of the electrons.

Actually, in the different attempts to calculate electron correlation (cf. Ref. 2) there was discovered a substantial effect of electron correlation on many metal properties. However, all these attempts were based on the use of a microscopic picture of the metals, and, in fact, were aimed at solving the many-body problem. The absence of a small parameter from these theories, due in the first place on the equal order of magnitude of the kinetic and potential energies of the electron, does not make these theories very valuable. Therefore, it is of considerable interest to construct a phenomenological theory sufficiently complete to take into account electron correlation, and not restricted to the narrow framework of some assumption, often far from justified, which one has to make in attempting to construct a microscopic theory of many particles. Such a phenomenological theory is the theory of a Fermi liquid, formulated by Landau for a quantum liquid such as He^3.

The object of the present work is to extend the Landau theory to include a degenerate electron liquid.

Let us note that in the kinetic theory of metal electrons the correlation is not calculated in practice. The Boltzmann kinetic equation is usually used instead. Moreover, partial calculation of electron correlations originating at small distances and leading to collisions is carried out by means of the electron-electron collision integral. Actually, these collisions play a small role,[3] and therefore make a slight contribution to the correlation. On the other hand, calculation of the correlation due to the identity of the electron shows that an appreciable change results in the kinetic equations.[4] What is important is that the influence of correlation does not reduce merely to collisions. It is therefore necessary to formulate a kinetic equation for electrons correctly taking into account the correlation of the particles. It will be shown below that the Landau theory allows one to carry out a phenomenological calculation of electron correlation. As will become evident from the following arguments, the theory of a Fermi liquid, formulated in Ref. 1 presupposes a small radius of effective force; it becomes therefore necessary to examine the variations resulting from Coulomb forces acting between electrons at an infinite radius.

In Secs. 2 and 3 it is shown that distant forces are correctly described by the self-consistent equations of the electromagnetic field.

2. LANDAU THEORY OF FERMI LIQUID AND THE HARTREE-FOCK APPROXIMATION

In order to understand how to extend the Landau theory to the case of electron interaction we first compare this theory with an approximate examination of a Fermi liquid based on the Hartree-Fock approximation. For simplicity, this comparison is made for the kinetic equations in the linear approximation, i.e., for slight deviations from the homogeneous and isotropic equilibrium state. In this case the kinetic equations, obtained in Hartree-Fock approximation for a system of half-spin particles, has the following form:[4]

$$\frac{\partial\varphi}{\partial t}+\left\{\frac{\mathbf{p}}{m}-\frac{1}{2}\int d\mathbf{p}'\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)\frac{\partial f_0}{\partial\mathbf{p}'}\right\}\frac{\partial\varphi}{\partial\mathbf{r}}+\frac{\partial f_0}{\partial\mathbf{p}}\frac{\partial}{\partial\mathbf{r}}\left\{\frac{1}{2}\int d\mathbf{p}'\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)\varphi(\mathbf{r},\mathbf{p})-\int d\mathbf{p}'d\mathbf{r}'\varphi(\mathbf{r}',\mathbf{p}')U(\mathbf{r}-\mathbf{r}'\,|\,)\right\}=0, \tag{1}$$

$$\frac{\partial\sigma}{\partial t}+\left(\left\{\frac{\mathbf{p}}{m}-\frac{1}{2}\int d\mathbf{p}'\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)\frac{\partial f_0}{\partial\mathbf{p}'}\right\}\frac{\partial}{\partial\mathbf{r}}\right)\sigma+\left(\frac{\partial f_0}{\partial\mathbf{p}}\frac{\partial}{\partial\mathbf{r}}\right)\left\{\frac{1}{2}\int d\mathbf{p}'\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)\sigma(\mathbf{r},\mathbf{p}')=0. \tag{2}$$

Here f_0 is the distribution function of the equilibrium state, ϕ is the non-equilibrium addition to the distribution function, σ is the spin distribution function (it is assumed that in the equilibrium state $\sigma_0 = 0$), $U(r)$ is the interaction energy of two Fermi particles, and $\nu(k) = \int d\mathbf{r}\, U(r)\exp(i\mathbf{kr})$. The spin distribution function is defined here by the relation

$$\sigma(\mathbf{r},\mathbf{p})=\mathrm{Sp}_\sigma\,\hat{\sigma}\hat{n}(\mathbf{r},\mathbf{p}), \tag{3}$$

where $\hat{\sigma}$ is the spin operator and $\hat{n}$ the density matrix.

Equation (2) is of the same form as the corresponding equation of the Landau theory, but Eq. (1) is substantially different since it contains a term with the particle-interaction energy U, which makes Eq. (1) integral in coordinate space. Such a difference can be removed by assuming the radius of the force effective between particles of the Fermi liquid to be small in comparison with the characteristic dimensions of the spatial inhomogeneities. In the case of a Fermi liquid such as He^3, this requirement, imposed on the radius of action of the force, is known to be satisfied. Then

$$\int d\mathbf{p}'d\mathbf{r}'\varphi(\mathbf{r}',\mathbf{p}')U(|\mathbf{r}-\mathbf{r}'|)\approx\int d\mathbf{p}'\varphi(\mathbf{r},\mathbf{p}')\int d\mathbf{r}'U(\mathbf{r}')\equiv\nu(0)\int d\mathbf{p}'\varphi(\mathbf{r},\mathbf{p}'). \tag{4}$$

Using relation (4) we can transform Eq. (1) to the following form

$$\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial\mathbf{r}}\left\{\frac{\mathbf{p}}{m}+\frac{1}{2}\mathrm{Sp}_\sigma\frac{\partial}{\partial\mathbf{p}}\int d\mathbf{p}'\hat{f}(\mathbf{p},\mathbf{p}')f_0(\mathbf{p}'\right\}-\frac{\partial f_0}{\partial\mathbf{p}}\frac{\partial}{\partial\mathbf{r}}\frac{1}{2}\mathrm{Sp}_\sigma\int d\mathbf{p}'\hat{f}(\mathbf{p},\mathbf{p}')\varphi(\mathbf{r},\mathbf{p}')=0, \tag{5}$$

where

$$\hat{f}(\mathbf{p},\mathbf{p}')=\nu(0)-\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)(1+\hat{\sigma}\hat{\sigma}')/2. \tag{6}$$

If the form of the function $\hat{f}(\mathbf{p},\mathbf{p}')$ is not specified, we can say that Eq. (5) agrees with the corresponding equation of the Landau theory. In Ref. 1 the function $\hat{f}(\mathbf{p},\mathbf{p}')$ is defined as the forward scattering amplitude of two quasi-particles of a Fermi liquid taken with a negative sign. In the Hartree-Fock approximation, according to Eq. (6), this effectively turns out to be so, but only for the scattering amplitude calculated in Born approximation.

From the above it is clear that the Landau theory of a Fermi liquid does not take into account the effects possible in the case of a force with effective radius comparable with the characteristic dimensions of an inhomogeneous liquid. Therefore, that theory cannot be directly applied to the case of Coulomb interaction, which is significant for electrons.

3. ENERGY AS A FUNCTIONAL OF THE DISTRIBUTION FUNCTION IN THE CASE OF COULOMB INTERACTION OF PARTICLES

The characteristic feature of the Landau theory of a Fermi liquid, which distinguishes it from gas theory, is that, owing to considerable self-consistent interaction of the particles the energy of an individual particles depends on the state of the surrounding particles. This, naturally, makes the energy of a system of particles unequal to the sum of the energies of the individual particles, and makes it some function of the distribution function.

Actually, in the Landau theory the variation of the energy density of a system of particles for infinitely small variation of the distribution function has the form

$$\delta E(\mathbf{r})=\mathrm{Sp}_\sigma\int\varepsilon(\mathbf{p},\mathbf{r})\,\delta n(\mathbf{p},\mathbf{r})\,d\mathbf{p}. \tag{7}$$

This formula, in fact, defines the energy (Hamiltonian function) of a quasi-particle, which differs from a free particle owing to the self-consistent interaction with surrounding particles.

Further, an essential role is played in the theory of a Fermi liquid by the variation in $\epsilon(\mathbf{p}, \mathbf{r})$ due to the variation in n. In Ref. 1 it is assumed that

$$\delta\varepsilon(\mathbf{p}, \mathbf{r}) = \mathrm{Sp}_{\sigma'}\int \hat{f}(\mathbf{p}, \mathbf{p}')\,\delta n(\mathbf{p}', \mathbf{r})\,d\mathbf{p}'. \tag{8}$$

The second variational derivative of the energy density $\hat{f}(\mathbf{p}, \mathbf{p}')$ determines many properties of the liquid, and in particular, it modifies substantially the kinetic equation.

In the preceding section it was shown that it is not possible to use the Landau theory directly in the case of a force having a large effective radius. This is connected with the use of Eq. (8) in this theory. One can understand the limited scope of this equation, for example, by use of Hartree-Fock approximation. In the general case one can say that

$$\delta\varepsilon(\mathbf{p}, \mathbf{r}) = \mathrm{Sp}_{\sigma'}\int \hat{F}(\mathbf{p}, \mathbf{p}';\ \mathbf{r}, \mathbf{r}')\,\delta n(\mathbf{p}', \mathbf{r}')\,d\mathbf{p}'d\mathbf{r}'. \tag{9}$$

In other words, the second variational derivative of the energy density of a liquid with respect to the distribution function depends not only on $\mathbf{p}$ and $\mathbf{p}'$ and spins, but is also a function of the coordinates $\mathbf{r}$ and $\mathbf{r}'$.

In particular, if we use the Hartree self-consistent approximation for particles interacting in a central-force field with potential energy $U(|\mathbf{r} - \mathbf{r}'|)$, then the function $\hat{F}$ assumes the following form:

$$\hat{F}_X(\mathbf{p}, \mathbf{p}';\ \mathbf{r}, \mathbf{r}') = U(|\mathbf{r} - \mathbf{r}'|). \tag{10}$$

In the Hartree approximation we disregard correlation completely since the distribution function of the system is taken in the form of the products of the distribution functions of the individual particles. The difference $(\hat{F} - \hat{F}_X)$ is therefore entirely due to the effect of the correlation of the particles. In the general case little can be said about this quantity. Note, however, that the case of practical interest, as shown below, is the one in which the radius of correlation of the particles is much less than the distance over which the distribution function varies significantly. In this case it is possible to take

$$\hat{F}(\mathbf{p}, \mathbf{p}';\ \mathbf{r}, \mathbf{r}') - \hat{F}_X \approx \delta(\mathbf{r} - \mathbf{r}')\,\hat{f}_1(\mathbf{p}, \mathbf{p}'). \tag{11}$$

Correspondingly, Eq. (9) takes the following form

$$\delta\varepsilon(\mathbf{p}, \mathbf{r}) = \mathrm{Sp}_{\sigma'}\int U(|\mathbf{r} - \mathbf{r}'|)\,\delta n(\mathbf{p}', \mathbf{r}')\,d\mathbf{p}'d\mathbf{r}' + \mathrm{Sp}_{\sigma'}\int \hat{f}_1(\mathbf{p}, \mathbf{p}')\,\delta n(\mathbf{p}', \mathbf{r})\,d\mathbf{p}'. \tag{12}$$

If the forces have a small effective radius, then Eq. (2) almost agrees with Eq. (8) taken in the Landau theory. Since in the case of He^3 the effective radius of forces as well as the radius of correlation of particles are both of the order of the interatomic distances, it is clear from the above that the use of Eq. (8) for liquid He^3 is entirely valid.

We go now to the concrete case of electrons. It is clear that the term in Eq. (12) containing the energy $U(r)$, equal in our case to e^2/r, leads, in general, to substantial difference from Eq. (8) which is the usual one for the Landau theory. However, to use Eq. (12) it is first necessary to justify approximation (11). Therefore, we shall now examine the question of electron correlation.

The correlation of degenerate electrons is due, in particular, to the sameness of the particles. This correlation is taken into account, for example, in the Hartree-Fock approximation. It is therefore possible to say at once that the distance in which the correlation is significant is of the order of the interelectron distance. Further, for electrons, for example of a metal, an important correlation is that due to their interaction (force correlation). This correlation at large distances is similar to the Debye correlation of particles, for example in electrolytes, and at small distances it is connected with the strong repulsion between electrons.

An essential feature of this correlation, for electron densities prevailing in a real metal, is that the characteristic distance in which the correlation is significant proves to be not much more than the interatomic distance.[2] It is therefore possible to say that, depending on the anisotropy, approximation (11) and Eq. (12) can be used for metal electrons. In this last relation the effect of distant forces is taken into account in the term containing the interaction energy of a pair of electrons $U(r) = e^2/r$. Calculation of the near correlation gives the function $\hat{f}_1$. The explicit form of this function is unfortunately unknown.

Apparently, it is only possible to say that the relative contribution due to the function $\hat{f}_1$ is determined by the parameter $e^2/\hbar v_0$ where v_0 is the velocity of the electrons on the Fermi surface.

Let us note a particular feature of Eq. (12) in the case $U(r) = e^2/r$. The first term on the right side of Eq. (12), if we consider it literally, leads in this case to a divergent integral for uniform distributions. This is caused by the fact that in the analysis of the system of electrons it is necessary to introduce the background of positive charges which compensate for the charge of the electrons and which correspond to the electronic charge of the ions. Therefore, in the first term on the right side of Eq. (12) we must take δn to mean the deviation of the distribution function from its spatially-homogeneous value. Hence, it follows that for a homogeneous spatial distribution of electrons Eq. (12) actually does not differ from the corresponding equation of the Landau theory, and, in particular, our function $\hat{f}_1$ coincides with the function $\hat{f}$ of the Landau theory. This means that Landau's theory of a Fermi liquid as applied to a spatially homogeneous state is suitable also in the case of Coulomb interaction between particles. The difference due to Eq. (12) arises only in the case of spatially-inhomogeneous distribution.

4. KINETIC EQUATION FOR THE ELECTRONS OF A DEGENERATE ELECTRON LIQUID

Usually the Boltzmann kinetic equation is used for the electrons in a metal. In this case the Lorentz force, which occurs in that equation, is determined by the self-consistent electromagnetic field.* In other words, the electromagnetic field equations have the following forms:

$$\operatorname{curl}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0, \qquad \operatorname{div}\mathbf{E} = 4\pi e\,\mathrm{Sp}_\sigma \int \hat{n}\,d\mathbf{p} \qquad \operatorname{div}\mathbf{H} = 0, \qquad \operatorname{curl}\mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 4\pi e\,\mathrm{Sp}_\sigma \int \mathbf{v}\hat{n}\,d\mathbf{p}. \tag{13}$$

In these equations $\mathrm{Sp}_\sigma \hat{n}$ represents exactly that distribution function for which the Boltzmann equation is usually written.

Such a self-consistent approach neglecting the collision integral exactly corresponds to the Hartree approximation. In the preceding section where we examined the Coulomb interaction of electrons we arrived at the result that the Hartree approximation accurately described the long-range part of the interaction of particles. Therefore, it is clear that the longitudinal part of the Lorentz force should be determined by the self-consistent electromagnetic field. Obviously, the long-range part of the electron interaction due to the transverse field should be reckoned in an analagous manner. Consequently, the long-range part of the electron interaction is accurately taken into account by the usual self-consistent kinetic equation.

The difference from usual kinetic equation arises from the neighboring correlation of the particles described by the second term of the right side of Eq. (12). However, since this term is entirely similar to Eq. (8) usually used in the Landau theory, the difference between the kinetic energy of the electrons of a degenerate electron liquid and the Boltzmann kinetic equation is, in fact, contained in the kinetic equation of the Landau theory. Therefore, the sought-for kinetic equation can be obtained from the results of Ref. 1. Then, disregarding spin-orbit interaction, it is possible to write the following kinetic equation for the electrons of a degenerate Fermi liquid:

$$\begin{gathered}\frac{\partial \hat{n}}{\partial t} + \frac{1}{2}\left(\frac{\partial \varepsilon_1}{\partial_{\mathbf{p}}}\frac{\partial \hat{n}}{\partial \mathbf{r}} + \frac{\partial \hat{n}}{\partial \mathbf{r}}\frac{\partial \varepsilon_1}{\partial \mathbf{p}}\right) - \frac{1}{2}\left(\frac{\partial \varepsilon_1}{\partial \mathbf{r}}\frac{\partial \hat{n}}{\partial \mathbf{p}} + \frac{\partial \hat{n}}{\partial \mathbf{p}}\frac{\partial \varepsilon_1}{\partial \mathbf{r}}\right) \\ + e\mathbf{E}\frac{\partial \hat{n}}{\partial \mathbf{p}} + \frac{e}{2c}\left\{\left[\mathbf{v}\times\mathbf{H}\right]\frac{\partial \hat{n}}{\partial \mathbf{p}} + \frac{\partial n}{\partial \mathbf{p}}\left[\mathbf{v}\times\mathbf{H}\right]\right\} + \frac{\beta}{2\hbar}(\hat{\sigma}\mathbf{H}\hat{n} - \hat{n}\hat{\sigma}\mathbf{H}) = I(\hat{n}).\end{gathered} \tag{14}$$

Here $\hat{n}(\mathbf{p}, \mathbf{r})$ is the distribution function, which is a function of the coordinates and of the momenta and which is a matrix in spin space, σ is the operator of electron spin, β is the magnetic moment of the electron, $I(\hat{n})$ is the collision integral, and $\delta\epsilon_1$ differs from expression (12) by the absence of self-consistent term,

$$\delta\varepsilon_1 = \mathrm{Sp}_{\sigma'} \int \hat{f}(\mathbf{p}, \mathbf{p}')\,\delta n(\mathbf{p}', \mathbf{r})\,d\mathbf{p}' - \frac{1}{2}\beta\hat{\sigma}\mathbf{H}. \tag{15}$$

In Eq. (14) and also Eq. (13) the value of $\mathbf{v}$, the velocity of the particles, in fact remains undetermined. If we neglect the effect of the lattice and consider an isotropic electron liquid, then it is possible, according to Landau, to use the Gallilean relativity principle. Then

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*Such an approach, applied to electron plasma, is developed in the work of Vlasov;[5] for metal electrons this approach is extensively used in the theory of the anomalous skin effect.[6]

$$\mathbf{v} \equiv \frac{\partial \varepsilon}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} + \mathrm{Sp}_{\sigma'} \int \hat{f}(\mathbf{p}, \mathbf{p}') \frac{\partial \hat{n}}{\partial \mathbf{p}'} d\mathbf{p}', \tag{16}$$

where m is the mass of the free electron. However, in a real metal m may differ from the mass of the free electron. Furthermore, this quantity, as well as the function $\hat{f}(\mathbf{p}, \mathbf{p}')$, can depend in principle on the direction. In addition, the region of large electromagnetic fields, $\hat{f}(\mathbf{p}, \mathbf{p}')$ may also depend on the electromagnetic field.

In conclusion I wish to thank V. L. Ginzburg for his interest in this work.

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APPLICATION OF MATRIX POLYNOMIALS TO DETERMINE SCATTERING PHASES

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A formalism of invariant matrix polynomials $L^{s',s}_{\ell',\ell;J}$ is developed for systems of particles of arbitrary spin. A general method for calculating $L^{s',s}_{\ell',\ell;J}$ is found, and the polynomials for total spin 0, ½ and 1 are written out explicitly. Equations (3.2) — (3.7) make the expansion of any invariant operator in polynomials $L^{s',s}_{\ell',\ell;J}$ a simple matter. It is shown that the coefficients in such an expansion of the scattering matrix are directly related to the phase-shifts. Formulae are derived for calculating the phase-shifts to any order of perturbation theory. In many specific examples this method simplifies the calculation of phase-shifts.

1. INTRODUCTION

To make comparisons of various theories with experiment, one must deal either with cross-sections or with phase-shifts. Whenever possible the phase-shifts are to be preferred, since they express the properties of the scattering with maximum conciseness. For example, the absolute sign of a phase is highly significant, as it indicates a qualitative difference (attractive or repulsive interaction) between two processes which may have equal cross-sections.

There is no existing theory which describes satisfactorily the strong interactions. But there are several theories[1-3] which give a reasonable qualitative picture of some particular processes. To compare these theories with one another, it is also convenient to study the behavior of the phase-shifts which they predict.

The interaction operator is specified in different ways in different theories. It may be given as a scattering matrix, as in perturbation theory,[4] or as a combination of various spin-dependent scalar potentials, or as an integral operator, for example in the Tamm-Dancoff method.[6,7] To calculate the phases corresponding to given values of J, ℓ and s, one must expand the interaction operator in spin-dependent spherical harmonics $g_{J,M}^{\ell,s}(\mathbf{n})$, as for example in the papers of Tamm[6] and Zharkov.[8] But the lack of covariance of these harmonics makes the expansion very complicated. The problem is greatly simplified if one expands the interaction operator into the invariant matrix operators (2.6′) which are bilinear combinations of the $G_{J,M}^{\ell,s}(\mathbf{n})$. The superiority of the operators $L_{\ell',\ell;J}^{s',s}$ over the functions $G_{J,M}^{\ell,s}(\mathbf{n})$ is as great as that of the tensor notation $H_{\mu\nu}$ over the six components of the vectors **E** and **H** for writing down the Maxwell equations.

In addition, the use of these operators greatly simplifies the diagonalization of the S-matrix with respect to the spin s, and hence the specification of the scattering in terms of a minimum number of real phases. The situation is similar to that which arose in the work of Rohrlich and Eisenstein,[2] with the difference that we are here dealing with a known S-matrix and the problem of finding the phase-shifts is reduced to the solution of algebraic equations.

As a further generalization of the invariant polynomials (2.6′), one could introduce matrix polynomials depending on three or more unit vectors. This would simplify the description of inelastic collisions in which three or more particles are produced.

2. MATRIX POLYNOMIALS. EXPLICIT EXPRESSIONS

Let $<\lambda', \xi'; \mathbf{p}'|R|\mathbf{p}; \xi, \lambda>$ be an operator depending on two vectors **p** and **p′** and invariant under three-dimensional rotations and reflections of the coordinate-system. λ and λ′ are spin variables describing the state of the system before and after collison. ξ and ξ′ are sets of parameters describing the nature and the isotopic spin of incident and scattered particles. To eliminate the wave-functions of the initial and final states, we temporarily transform to an arbitrary representation g. Then the general principles of transformation theory[9] give

$$\langle\lambda', \xi'; \mathbf{p}'|R|\mathbf{p}; \xi, \lambda\rangle = \sum_{g'g}\langle\lambda', \xi'; \mathbf{p}'|g'\rangle\langle g'|R|g\rangle\langle g|\mathbf{p}; \xi, \lambda\rangle. \tag{2.1}$$

Separating the spin and angle dependence from the radial dependence of the wave-functions $<\lambda', \xi'; \mathbf{p}'|$ and $|\mathbf{p}; \xi, \lambda>$, we have

$$\langle\lambda', \xi'; \mathbf{p}'| = \sum_{J'M'l's'}\langle\xi', p'; J', M', s', l'|\langle\lambda', \mathbf{n}'; J', M', s', l'| \tag{2.2}$$

$$|\mathbf{p}; \xi, \lambda\rangle = \sum_{JMls}|l, s, M, J; p, \xi\rangle|l, s, M, J; \mathbf{n}, \lambda\rangle, \tag{2.2'}$$

where

$$|l, s, M, J; \mathbf{n}, \lambda\rangle = (g_{J,M}^{l,s})^{+}(\mathbf{n}, \lambda),\ \langle\lambda, \mathbf{n}; J, M, s, l| = g_{J,M}^{l,s}(\mathbf{n}, \lambda)$$

are eigenfunctions of the operators J^2, J_z, L^2 and S^2. Substituting Eq. (2.2) and (2.2′) into (2.1) and summing over g and g′, we obtain

$$\langle\lambda', \xi'; \mathbf{p}'|R|\mathbf{p}; \xi, \lambda\rangle = \sum_{J'M's'l'}\sum_{JMsl}\langle\xi', p'; J', M', s', l'|R|l, s, M, J; p, \xi\rangle \times\langle\lambda', \mathbf{n}'; J', M', s', l'|l, s, M, J; \mathbf{n}, \lambda\rangle. \tag{2.3}$$

Because of the rotation-invariance, J and M are conserved; the matrix elements of R which appear in the sum are diagonal in J and M and depend on M only through a factor $\delta_{M'M}$. Thus

$$\langle\xi', p'; J', M', s', l'|R|l, s, M, J; p, \xi\rangle = \delta_{J'J}\delta_{M'M}<\xi', p'; s', l'|R_J|l, s; p, \xi\rangle, \tag{2.4}$$

so that

$$\langle\lambda', \xi'; \mathbf{p}'|R|\mathbf{p}; \xi, \lambda\rangle = \sum_{J}\sum_{s's}\sum_{l'l}\langle\xi', p'; s', l'|R_J|l, s; p, \xi\rangle\sum_{M}\langle\lambda', \mathbf{n}'; J, M, s', l'|l, s, M, J; \mathbf{n}, \lambda\rangle. \tag{2.5}$$

The reflection-invariance of R restricts the summation over ℓ and ℓ' in Eq. (2.5) to values which conserve the parity π. Thus Eq. (2.5) is precisely the expansion of the operator R into generalized matrix polynomials[10] of the form

$$\langle\lambda', \mathbf{n}'; J, s', l' | l, s, J; \mathbf{n}, \lambda\rangle = \sum_M \langle\lambda', \mathbf{n}'; J, M, s', l' | l, s, M, J; \mathbf{n}, \lambda\rangle \tag{2.6}$$

or in a different notation*

$$L^{s',s}_{l';l;J}(\lambda', \mathbf{n}'; \lambda, \mathbf{n}) = 4\pi \sum_M g^{l',s'}_{J,M}(\mathbf{n}', \lambda')(g^{l,s}_{J,M})^+(\mathbf{n},\lambda). \tag{2.6'}$$

Using the orthogonality relations of tensor harmonics, we can show that

$$\int L^{\alpha,\beta}_{lm;J}(\mathbf{n}, \mathbf{n}') L^{\gamma,\delta}_{np;J_0}(\mathbf{n}', \mathbf{n}_0)\, d\mathbf{n}' = 4\pi\delta_{JJ_0}\delta_{mn}\delta_{\beta\gamma} L^{\alpha,\delta}_{l,p;J}(\mathbf{n}, \mathbf{n}_0). \tag{2.7}$$

We now consider the explicit calculation of the matrix polynomials (2.6′). First we express $g^{\ell,s}_{J,M}$ in terms of angle and spin functions $Y_{\ell,m}$ and $\chi_{s,\mu}$, according to the rules for transforming from the J, M, s, ℓ – representation to the ℓ, m, s, μ – representation. This gives

$$g^{l,s}_{J,M}(\mathbf{n}, \lambda) = \sum_{m,\mu} \langle J, M, s, l | l, m; s, \mu\rangle Y_{l,m}(\mathbf{n}) \chi_{s,\mu}(\lambda), \tag{2.8}$$

where $< J, M, s, \ell \mid \ell, m; s, \mu >$ is a Clebsch-Gordan coefficient.[11] In the representation with S^2 and S_z diagonal, $\chi_{s,\mu}(\lambda) = \delta_{\mu\lambda}$ and so

$$\begin{aligned} L^{s',s}_{l',l;J}(\mathbf{n}', \lambda'; \mathbf{n}, \lambda) = 4\pi \sum_{\mu'\mu} \delta_{\mu'\lambda'}\delta_{\mu\lambda} \sum_M \langle J, M; s', l' | l'; M-\mu'; s', \mu'\rangle \\ \times \langle J, M; s, l | l, M-\mu; s, \mu\rangle Y_{l',M-\mu'}(\mathbf{n}') Y^*_{l,M-\mu}(\mathbf{n}). \end{aligned} \tag{2.9}$$

At first glance it would appear convenient to reduce the triple sum to a sum over M alone by using the factors $\delta_{\mu\lambda}$. But this is not the best way to proceed. It would give us the values of particular matrix elements with known λ and λ', but it would not allow us to express $L^{s',s}_{\ell',\ell;J}$ in closed form as a sum of vector spin operators[12] which reduce in the case $s' = s$ to ordinary spin matrices. So we keep the double sum over μ' and μ, and express $\delta_{\mu'\lambda'}\delta_{\mu\lambda}$ by means of $(2s' + 1)(2s + 1)$ combinations of the three components of the relevant vector operators. The summation over M can be performed by using recursive relations of the form†

$$\sqrt{(l \pm m + n_1)(l \pm (\mp m) + n_2)}\, Y_{l+i_1, m+i_2}(\mathbf{n}) = D(l; n, i) Y_{lm}(\mathbf{n}), \tag{2.10}$$

where n, i are integers, and $D(\ell; n, i)$ is a differential operator independent of m and operating upon $Y_{\ell,m}(\mathbf{n})$. Having chosen the appropriate values of n, i, we avoid the appearance of Clebsch-Gordan coefficients involving m, and we can complete the summation by using the addition theorem for spherical harmonics

$$(2l + 1) P_l(\mathbf{n}'\mathbf{n}) / 4\pi = \sum_m Y_{l,m}(\mathbf{n}') Y^*_{l,m}(\mathbf{n}). \tag{2.11}$$

Considering only processes in which the spin is not changed in the collision, we exhibit the explicit forms of the matrix polynomials for S = 0, ½, 1. These are the cases of greatest physical interest.

S = 0. The polynomials $L^{0,0}_{\ell,\ell;\ell}$ describe the scattering of two spinless particles or the singlet scattering of two particles with spin. They are identical up to a factor with Legendre polynomials

$$L^{0,0}_{l,l;l}(\mathbf{n}', \mathbf{n}) = 4\pi \sum_m Y_{l,m}(\mathbf{n}') Y^*_{l,m}(\mathbf{n}) = (2l + 1) P_l(\mathbf{n}'\mathbf{n}). \tag{2.12}$$

S = ½. The polynomials $L^{1/2,1/2}_{\ell',\ell;J}$ were first introduced by Tamm and collaborators[3] and have been applied extensively to the scattering of π-mesons by nucleons.[13,14] Transitions $\ell \rightarrow \ell' = \ell \pm 1$ are forbidden by parity-conservation, so that only the two polynomials $L^{1/2,\,1/2}_{J-1/2,\,J-1/2;\,J}$ and $L^{1/2,\,1/2}_{J+1/2,\,J+1/2;\,J}$ are of interest. We derive the explicit form of the first of these from Eq. (2.9). Using the known values of the Clebsch-Gordan coefficients,[11] we find

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*The factor 4π is put in so as to eliminate factors of $(1/4\pi)$ from the explicit forms of the polynomials.

†The sign $\pm(\mp m)$ means that all four combinations of sign are allowed for the two m's.

$$L^{1/2,\,1/2}_{J-1/2,\,J-1/2,\,J}(\mathbf{n}',\lambda';\mathbf{n},\lambda)=\frac{4\pi}{2J}\Big\{\sum_M (J+M)\,Y_{J-1/2,M-1/2}(\mathbf{n}')\,Y^*_{J-1/2,M-1/2}(\mathbf{n})\,\delta_{1/2\lambda'}\delta_{1/2\lambda}$$

$$+\sum_M\sqrt{(J+M)(J-M)}\,Y_{J-1/2,\,M-1/2}(\mathbf{n}')\,Y^*_{J-1/2,\,M+1/2}(\mathbf{n})\,\delta_{1/2\lambda'}\,\delta_{-1/2\lambda} \tag{2.13}$$

$$+\sum_M\sqrt{(J-M)(J+M)}\,Y_{J-1/2,\,M+1/2}(\mathbf{n}')\,Y^*_{J-1/2,\,M-1/2}(\mathbf{n})\,\delta_{-1/2\lambda'}\,\delta_{1/2\lambda}+\sum_M (J-M)\,Y_{J-1/2,\,M+1/2}(\mathbf{n}')\,Y_{J-1/2,\,M+1/2}(\mathbf{n})\,\delta_{-1/2\lambda'}\,\delta_{-1/2\lambda}\Big\}$$

Transforming from J, M to $\ell = J - 1/2$, $m = M - 1/2$, and introducing spin matrices by

$$\delta_{1/2\lambda'}\delta_{1/2\lambda}=1/2\,(1+\sigma_z)_{\lambda'\lambda};\ \ \delta_{1/2\lambda'}\,\delta_{-1/2\lambda}=1/2\,(\sigma_x+i\sigma_y)_{\lambda'\lambda};\ \ \ \delta_{-1/2\lambda'}\,\delta_{1/2\lambda}=1/2\,(\sigma_x-i\sigma_y)_{\lambda'\lambda};\ \delta_{-1/2\lambda'}\,\delta_{-1/2\lambda}=1/2\,(1-\sigma_z)_{\lambda'\lambda}; \tag{2.14}$$

we obtain

$$L^{1/2,\,1/2}_{J-1/2,\,J-1/2;\,J}(\mathbf{n}',\mathbf{n})=(2\pi/(2l+1))\Big\{\sum_m (l+m+1)\,Y_{l,\,m}(\mathbf{n}')\,Y^*_{l,m}(\mathbf{n})\,(1+\sigma_z)$$

$$+\sum_m\sqrt{(l+m)(l-m+1)}\,Y_{l,\,m-1}(\mathbf{n}')\,Y^*_{l,\,m}(\mathbf{n})\,(\sigma_x+i\sigma_y)+\sum_m\sqrt{(l-m)(l+m+1)}\,Y_{l,\,m+1}(\mathbf{n}')\,Y^*_{l,\,m}(\mathbf{n})\,(\sigma_x-i\sigma_y)$$

$$+\sum_m (l-m+1)\,Y_{l,\,m}(\mathbf{n}')\,Y^*_{l,\,m}(\mathbf{n})\,(1-\sigma_z)\Big\}\,. \tag{2.15}$$

The summation over m is done by using the recursion formulae

$$(l\pm m+1)\,Y_{l,\,m}=(l+1)\,Y_{l,\,m}\mp i\partial Y_{l,\,m}/\partial\varphi; \tag{2.16}$$

$$\sqrt{(l\pm m)(l\mp m+1)}\,Y_{l,\,m-1}=e^{\mp i\varphi}(i\cot\theta\,\partial Y_{l,\,m}/\partial\varphi\pm\sin\theta\,Y\partial_{l,\,m}/\partial\cos\theta). \tag{2.17}$$

We use also

$$\mathbf{n}'\mathbf{n}=\cos\theta'\cos\theta+\sin\theta'\sin\theta\cos(\varphi'-\varphi), \tag{2.18}$$

and obtain finally*

$$L^{1/2,\,1/2}_{J-1/2,\,J-1/2;\,J}(\mathbf{n}',\ \mathbf{n})=(l+1)\,P_l-i\sigma\,[\mathbf{n}'\mathbf{n}]\,P'_l,\ l=J-1/2,\ J=1,2,3,\ldots \tag{2.19}$$

A similar calculation gives

$$L^{1/2,\,1/2}_{J+1/2,\,J+1/2,\,J}(\mathbf{n}',\mathbf{n})=lP_l+i\sigma\,[\mathbf{n}'\mathbf{n}]\,P'_l,\ l=J+1/2,\ \ J=0,\ 1,\ 2,\ldots \tag{2.20}$$

S = 1. The matrix polynomials $L^{1,\,1}_{\ell',\,\ell;\,J}$ greatly facilitate the analysis of the interaction of two fermions in the triplet state. Becasue of parity-conservation, the four polynomials $L^{1,\,1}_{J+1,\,J;\,J}$ $L^{1,\,1}_{J,\,J+1;\,J}$ do not appear in the expansion of R. Out of nine possible polynomials, only five have physical interest, three diagonal in ℓ

$$L^{1,1}_{J-1,\,J-1;\,J}\ (\mathbf{n}',\mathbf{n})\equiv L_{11,\,J}\ (\mathbf{n}',\mathbf{n}),\quad L^{1,1}_{J,\,J;\,J}(\mathbf{n}',\mathbf{n})\equiv L_{22,\,J}(\mathbf{n}',\mathbf{n}),\ \ L^{1,1}_{J+1,\,J+1;\,J}\ (\mathbf{n}',\mathbf{n})\equiv L_{33,\,J}(\mathbf{n}',\mathbf{n})$$

and three non-diagonal

$$L^{11}_{J-1,\,J+1;\,J}(\mathbf{n}',\ \mathbf{n})\equiv L_{13\,J}(\mathbf{n}',\ \mathbf{n});\ \ L^{11}_{J+1,\,J-1;\,J}(\mathbf{n}',\ \mathbf{n})\equiv L_{31,\,J}(\mathbf{n}',\ \mathbf{n}),$$

Also $L_{31,J}(\mathbf{n}',\mathbf{n}) = [L_{13,J}(\mathbf{n},\mathbf{n}')]^+$. As in the case $S = 1/2$, we start from Eq. (2.9). Since λ' and λ now take three values, we must introduce instead of σ_x, σ_y, σ_z, the spin operators for S = 1,

$$S_x=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix};\quad S_y=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix};\quad S_z=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

---

*In Ref. 3 [Eq. (4.6)], Ref. 14 [Eq. (5.16)], and Ref. 13 [Eq. (5.12)], there are misprints in the expressions for $L^{\pm}_{\ell}$. Either $\sigma[\mathbf{n}'\mathbf{n}]$ should be divided by $\sin\theta$, or $P^1_\ell(\cos\theta)$ should be changed to P'_ℓ (cos

Then

$$\partial_{1\lambda'}\partial_{1\lambda}=\frac{1}{2}S_z(S_z+1);\ \partial_{1\lambda'}\partial_{0\lambda}=\frac{1}{\sqrt{2}}S_z(S_x+iS_y);\ \delta_{1\lambda,}\partial_{--1\lambda}=\frac{1}{2}(S_x+iS_y)^2;$$
$$\partial_{0\lambda'}\partial_{1\lambda}=\frac{1}{\sqrt{2}}(S_x-iS_y)S_z;\ \partial_{0\lambda'}\partial_{0\lambda}=1-S_z^2;\ \delta_{0\lambda'}\delta_{-1\lambda}=-\frac{1}{\sqrt{2}}(S_x+iS_y)S_z; \tag{2.21}$$
$$\partial_{-1\lambda'}\partial_{1\lambda}=\frac{1}{2}(S_x-iS_y)^2;\ \partial_{-1\lambda'}\delta_{0\lambda}=-\frac{1}{\sqrt{2}}S_z(S_x-iS_y);\ \partial_{-1\lambda'}\partial_{-1\lambda}=\frac{1}{2}S_z(S_z-1).$$

In addition to Eq. (2.16), (2.17), the summation over m requires four more recursion formulae,

$$\sqrt{(l\pm m)(l\pm m-1)}\,Y_{l-1,\,m\mp1}=\sqrt{\frac{2l-1}{2l+1}}\,e^{\mp i\varphi}\left(\pm l\sin\theta\,Y_{l,\,m}+\frac{i}{\sin\theta}\frac{\partial Y_{l,\,m}}{\partial\varphi}+\sin\theta\cos\theta\frac{\partial Y_{l,\,m}}{\partial\cos\theta}\right); \tag{2.22}$$

$$\sqrt{(l\mp m+1)(l\mp m+2)}\,Y_{l+1,\,m\mp1}=\sqrt{\frac{2l+3}{2l+1}}e^{\mp i\varphi}\left(\pm(l+1)\sin\theta\,Y_{l,m}+\frac{i}{\sin\theta}\frac{\partial Y_{l,m}}{\partial\varphi}\pm\sin\theta\cos\theta\frac{\partial Y_{l,\,m}}{\partial\cos\theta}\right). \tag{2.23}$$

Omitting the lengthy intermediate steps, we exhibit the final expressions for all five polynomials,

$$L_{11,\,J}(\mathbf{n}',\ \mathbf{n})=\frac{1}{l+1}\{(l+1)P_l+[2(\mathbf{n}'\mathbf{n})+(l+1)(\mathbf{Sn})(\mathbf{Sn}')-(l+2)(\mathbf{Sn}')(\mathbf{Sn})]P_l'-(\mathbf{S}[\mathbf{n}'\mathbf{n}])^2P_l''\}; \tag{2.24}$$

$$L_{22,\,J}(\mathbf{n}',\ \mathbf{n})=\frac{2l+1}{l(l+1)}\{l(l+1)P_l+[-2(\mathbf{n}'\mathbf{n})+(\mathbf{Sn}')(\mathbf{Sn})]P_l'+(\mathbf{S}[\mathbf{n}'\mathbf{n}])^2P_l''\}; \tag{2.25}$$

$$L_{33,\,J}(\mathbf{n}',\ \mathbf{n})=\frac{1}{l}\{-lP_l+[2(\mathbf{n}'\mathbf{n})-l(\mathbf{Sn})(\mathbf{Sn}')+(l-1)(\mathbf{Sn}')(\mathbf{Sn})]P_l'-(\mathbf{S}[\mathbf{n}'\mathbf{n}])^2P_l''\}; \tag{2.26}$$

$$\begin{aligned}L_{13,\,J}(\mathbf{n}',\ \mathbf{n})=\frac{1}{\sqrt{l(l-1)}}\{l[-2l+(2l-1)(\mathbf{Sn}')^2]P_l+[2(\mathbf{n}'\mathbf{n})\\-(2l-1)(\mathbf{n}'\mathbf{n})(\mathbf{Sn}')^2+(l-1)((\mathbf{Sn})(\mathbf{Sn}')+(\mathbf{Sn}')(\mathbf{Sn}))]P_l'-(\mathbf{S}[\mathbf{n}'\mathbf{n}])^2P_l''\};\end{aligned} \tag{2.27}$$

$$\begin{aligned}L_{31,\,J}(\mathbf{n}',\mathbf{n})=\frac{1}{\sqrt{(l+1)(l+2)}}\{(l+1)[-2(l+1)+(2l+3)(\mathbf{Sn}')^2]P_l+[2(\mathbf{n}'\mathbf{n})\\+(2l+3)(\mathbf{n}'\mathbf{n})(\mathbf{Sn}')^2-(l+2)((\mathbf{Sn})(\mathbf{Sn}')+(\mathbf{Sn}')(\mathbf{Sn}))]P_l'-(\mathbf{S}[\mathbf{n}'\mathbf{n}])^2P_l''\}.\end{aligned} \tag{2.28}$$

Similar polynomials for $S = 1$ were obtained by Ritus[15] using a somewhat different method.

3. EXPANSION OF INVARIANT OPERATORS INTO MATRIX POLYNOMIALS

The preceding formulae show that the $L^{1,1}_{\ell',\ell;\,J}$ have a complicated form, and it is a laborious task to expand an operator in these polynomials by ordinary methods. However, the requirements of rotation and reflection invariance restrict the possible form of an operator. The most general invariant operator R is in fact*

$$R=V_1+(\mathbf{Sn})^2V_2+(\mathbf{Sn}')^2V_3+(\mathbf{Sn}')(\mathbf{Sn})V_4+(\mathbf{Sn})(\mathbf{Sn}')V_5, \tag{3.1}$$

where $V_i = V_i(\mathbf{n}'\mathbf{n})$ are arbitrary functions of $\mathbf{n}'\mathbf{n}$. The problem of finding the expansion coefficients is greatly simplified by using the following identities which follow from Eq. (2.24) – (2.28),

$$(2J+1)P_J=L_{11,\,J+1}+L_{22,\,J}+L_{33,\,J-1}; \tag{3.2}$$

$$(2J+1)(\mathbf{n}'\mathbf{n})P_J=\frac{J+1}{2J+3}L_{11,\,J+2}+\frac{J}{2J-1}L_{11,\,J}+\frac{J+1}{2J+3}L_{22,\,J+1}+\frac{J}{2J-1}L_{22,\,J-1}+\frac{J+1}{2J+3}L_{33,\,J}+\frac{J}{2J-1}L_{33,\,J-2}; \tag{3.3}$$

$$(2J+1)(\mathbf{Sn})^2P_J=\frac{J+2}{2J+3}L_{11,\,J+1}+L_{22,\,J}+\frac{J-1}{2J-1}L_{33,\,J-1}+\frac{\sqrt{(J+1)(J+2)}}{2J+3}L_{13,\,J+1}+\frac{\sqrt{J(J-1)}}{2J-1}L_{31,\,J-1}; \tag{3.4}$$

$$(2J+1)(\mathbf{Sn}')^2P_J=\frac{J+2}{2J+3}L_{11,\,J+1}+L_{22,\,J}+\frac{J-1}{2J-1}L_{33,\,J-1}+\frac{\sqrt{J(J-1)}}{2J-1}L_{13,\,J-1}+\frac{\sqrt{(J+1)(J+2)}}{2J+3}L_{31,\,J+1}; \tag{3.5}$$

$$\begin{aligned}(2J+1)(\mathbf{Sn}')(\mathbf{Sn})P_J=\frac{J+1}{2J+1}L_{11,\,J}+\frac{J-1}{2J-1}L_{22,\,J-1}+\frac{J+2}{2J+3}L_{22,\,J+1}\\+\frac{J}{2J+1}L_{33,\,J}+\frac{\sqrt{J(J+1)}}{2J+1}L_{13,\,J}+\frac{\sqrt{J(J+1)}}{2J+1}L_{31,\,J};\end{aligned} \tag{3.6}$$

---

*The invariant combination $i\mathbf{S}[\mathbf{n}'\mathbf{n}]$ is contained in the fourth and fifth terms by $i\mathbf{S}[\mathbf{n}'\mathbf{n}] = (\mathbf{Sn}')(\mathbf{Sn}) - (\mathbf{Sn})(\mathbf{Sn}')$, and $(\mathbf{S}'[\mathbf{n}'\mathbf{n}])^2$ is contained in all five by $(\mathbf{S}[\mathbf{n}'\mathbf{n}])2 = 2[1-(\mathbf{n}'\mathbf{n})^2]-(\mathbf{Sn})^2-(\mathbf{Sn}')^2+(\mathbf{n}'\mathbf{n})[(\mathbf{Sn})(\mathbf{Sn}')+(\mathbf{Sn}')(\mathbf{Sn})]$.

$$(2J+1)(\mathrm{Sn})(\mathrm{Sn}')P_J=\frac{J+1}{2J+3}L_{11,\,J+2}+\frac{2J}{(2J+1)(2J-1)}L_{11,\,J}+\frac{J}{2J-1}L_{22,\,J-1}$$
$$+\frac{J+1}{2J+3}L_{22,\,J+1}-\frac{2(J+1)}{(2J+1)(2J+3)}L_{33,\,J}+\frac{J}{2J-1}L_{33,\,J-2}+\frac{\sqrt{J(J+1)}}{2J+1}L_{13,\,J}+\frac{\sqrt{J(J+1)}}{2J+1}L_{31,\,J}. \quad (3.7)$$

Sometimes we encounter operators which contain not only the P_ℓ but also their derivatives P'_ℓ and P''_ℓ. Terms containing P'_ℓ and P''_ℓ can be expressed as simple combinations of matrix polynomials $L^{1,1}_{\ell',\ell;\,J}$ similar to Eq. (3.2) —(3.7), but for reasons of space we do not here go into the details.

4. DETERMINATION OF PHASE SHIFTS FROM THE SCATTERING MATRIX

The S-matrix is as a rule calculated using quantum field theory, in which for a variety of reasons it is convenient to work in the p-representation. Thus the scattering matrix is first obtained as an operator depending on the initial and final moment $\mathbf{p}$ and $\mathbf{p}'$ and on the parameters λ, ξ, λ', ξ'. To find the phases we transform to the J, M, s, ℓ, p, ξ – representation,

$$\langle\lambda',\,\xi';\,\mathbf{p}'|S|\mathbf{p};\,\xi,\,\lambda\rangle=\sum_{J_2M_2s_2l_2}\sum_{J_1M_1s_1l_1}\int\langle\lambda',\,\xi';\,\mathbf{p}'|J_2,\,M_2,\,s_2,\,l_2,\,p_2,\,\xi_2\rangle \quad (4.1)$$
$$\times\, p_2^2dp_2\langle\xi_2,\,p_2;\,l_2,\,s_2,\,M_2,\,J_2|S|J_1,\,M_1,\,s_1,\,l_1;\,p_1,\,\xi_1\rangle p_1^2dp_1\langle\xi_1,\,p_1;\,l_1,\,s_1,\,M_1,\,J_1|\mathbf{p};\,\xi,\,\lambda\rangle,$$

with the transformation-functions

$$\langle\lambda,\,\xi;\,\mathbf{n},\,p|J,\,M,\,s,\,l,\,p',\,\xi'\rangle=\langle\lambda,\,\mathbf{n}|J,\,M,\,s,\,l\rangle\langle\xi,\,p|p',\,\xi'\rangle, \quad (4.2)$$

$$\langle\lambda,\,\mathbf{n}|J,\,M,\,s,\,l\rangle=g^{l,s}_{J,M}(\mathbf{n},\,\lambda), \quad (4.3)$$

$$\langle\xi,\,p|p',\,\xi'\rangle=(1/p)\,\delta(p-p')\,\delta_{\xi'\xi} \quad (4.4)$$

normalized by the conditions

$$\sum_{JMsl}\langle\lambda',\,\mathbf{n}'|J,M,s,l\rangle\langle J,M,s,l|\mathbf{n},\lambda\rangle=\delta_{\lambda'\lambda}\delta(\mathbf{n}'-\mathbf{n}), \quad (4.5)$$

$$\sum_{\lambda}\int\langle l',\,s',\,M',\,J'|\mathbf{n},\,\lambda\rangle\,d\mathbf{n}\,\langle\lambda,\,\mathbf{n}|J,\,M,\,s,\,l\rangle=\delta_{JJ'}\,\delta_{MM'}\,\delta_{ss'}\,\delta_{ll'}, \quad (4.6)$$

$$\sum_{\xi_1}\int\langle\xi',\,p'|p_1,\,\xi_1\rangle\,p_1^2dp_1\,\langle\xi_1,\,p_1|p,\,\xi\rangle=\delta_{\xi'\xi}\,\delta(p'-p). \quad (4.7)$$

If we compare the expansion of the S-matrix in matrix polynomials $L^{s',s}_{\ell',\ell;\,J}(\mathbf{n}',\lambda';\mathbf{n},\lambda)$

$$\langle\lambda',\xi';\,\mathbf{p}'|S|\mathbf{p};\,\xi,\,\lambda\rangle=\sum_{s's}\sum_{l'l}\alpha^{s's}_{l',\,l;\,J}(p',\,\xi';\,p,\,\xi)\,L^{s's}_{l',\,l;\,J}\;(\mathbf{n}',\,\lambda';\,\mathbf{n},\,\lambda) \quad (4.8)$$

with the expansion (4.1), we see that the expansion coefficients $\alpha^{s',s}_{\ell',\ell;\,J}$ are related to the J, M, s, ℓ, p, ξ —representation of the S-matrix by

$$4\pi\delta_{J'J}\delta_{M'M}p'p\alpha^{s's}_{l',\,l;\,J}(p',\,\xi';\,p,\,\xi)=\langle\xi',\,p';\,J',\,M',\,s',\,l'|S|l,\,s,\,M,\,J;\,p,\,\xi\rangle \quad (4.9)$$

or

$$4\pi p'p\alpha^{s',s}_{l',l;\,J}(p',\,\xi';\,p,\,\xi)=\langle\xi',\,p';\,s',\,l'|S|l,\,s;\,p,\,\xi\rangle. \quad (4.10)$$

To introduce the phase-shifts, we use two fundamental properties of the S-matrix; it must be unitary and symmetric. The first property follows from the normalization to unity of the flux of incident and scattered particles, or from the conservation of the number of particles. The second property follows from the time-reversibility of the theory, which is connected with the principle of detailed balance. Since the S-matrix is unitary, it can always be written in the form

$$S=\exp(2iQ), \quad (4.11)$$

where Q is a hermitian phase-matrix. Then, since Q is hermitian, it can be reduced by a unitary transformation U to a diagonal matrix Q_0 with N real diagonal elements*

$$Q=U^{-1}Q_0U. \quad (4.12$$

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*Here N is the number of different sets of s, ℓ, ξ, corresponding to a given value of J.

But S is symmetric, i.e. $S^+ = S^*$, and therefore Q and U are real. The orthogonality of U imposes $\frac{1}{2}N(N+1)$ additional conditions, so that only $\frac{1}{2}N(N+1)$ out of the original $2N^2$ real parameters are independent. Of these, $\frac{1}{2}N(N-1)$ are still to some extent arbitrary, since they are connected with the matrix U and depend on the choice of representation. Only the N diagonal elements, which are the actual phase shifts, express essential properties of the interaction. Thus the problem of calculating phase-shifts reduces to the diagonalization of the matrix α, and the elements of the diagonalized matrix α' are related to the phases by

$$4\pi p' p \alpha_0^{(k)}(p', \; p) = e^{2i\delta_k}, \quad k = 1, 2, \ldots, N; \tag{4.13}$$

or by

$$\tan 2\delta_k = \operatorname{Im} \alpha_0^{(k)} / \operatorname{Re} \alpha_0^{(k)}. \tag{4.14}$$

We often deal not with the S-matrix itself but with $R = S - 1$. If we denote by $a^{S',S}_{\ell',\ell;\,J}$ the coefficients in the expansion of R into polynomials $L^{S',S}_{\ell',\ell;\,J}$, then we have instead of Eq. (4.13), (4.14),

$$-2\pi i p' p a_0^{(k)}(p', \; p) = e^{i\delta_k} \sin \delta_k \tag{4.15}$$

$$\tan \delta_k = -\operatorname{Re} a_0^{(k)} / \operatorname{Im} a_0^{(k)}. \tag{4.16}$$

Eq. (4.14) and (4.16) do not quite solve the problem, because they are obtained by assuming the exact unitarity of the S-matrix, which ensures the compatibility of the real equations which follow from the complex equations (4.13) and (4.15). But in practice the S-matrix is often given as a finite sum which is unitary only up to the order of the last term included in the sum. Then it can happen, for example, that the equations

$$-2\pi p' p \operatorname{Re} a_{(k)} = \sin^2 \delta_k; \tag{4.17}$$

$$2\pi p' p \operatorname{Im} a_{(k)} = \sin \delta_k \cos \delta_k \tag{4.18}$$

are incompatible, and Eqs. (4.14) and (4.16) may then introduce errors which of lower order than the terms neglected in the S-matrix. We now derive a general expression for the phase-shift which preserves the order of accuracy with which the S-matrix is given. The S-matrix can be written in the form

$$S = 1 + \sum_{n=1}^{\infty} g^n C_n; \tag{4.19}$$

$$C_n = \frac{(-i)^n}{g^n \, n!} \int_{-\infty}^{\infty} dx_n \ldots \int_{-\infty}^{\infty} dx_1 P\,(H(x_n) \ldots H(x_1)). \tag{4.20}$$

The coefficients in the expansion of R into polynomials $L^{S',S}_{\ell',\ell;\,J}$ are then

$$a^{s',s}_{l',l;\,J} = \sum_{n=1}^{\infty} g^n b^{s',s}_{l',l;\,J}(n). \tag{4.21}$$

After diagonalization with respect to s and ℓ, Eq. (4.17) and (4.18) give*

$$2\pi p^2 \sum_{n=1}^{\infty} g^n \operatorname{Re} b(n) = -\sin^2 \delta; \tag{4.22}$$

$$2\pi p^2 \sum_{n=1}^{\infty} g^n \operatorname{Im} b(n) = \sin \delta \cos \delta, \tag{4.23}$$

hence

$$|\sin \delta| = 2\pi p^2 \left[\left(\sum_{n=1}^{\infty} g^n \operatorname{Re} b(n) \right)^2 + \left(\sum_{n=1}^{\infty} g^n \operatorname{Im} b(n) \right)^2 \right]^{1/2} = 2\pi p^2 \left[\sum_{n=2}^{\infty} g^n f_n \right]^{1/2}, \tag{4.24}$$

where

$$f_{2m} = 2 \sum_{k=1}^{m-1} (\operatorname{Re} b(k) \operatorname{Re} b(2m-k) + \operatorname{Im} b(k) \operatorname{Im} b(2m-k)) + [\operatorname{Re} b(m)]^2 + [\operatorname{Im} b(m)]^2,$$

---

* Without loss of generality we put $p' = p$.

$$f_{2m+1} = 2\sum_{k=1}^{m} (\operatorname{Re} b(k) \operatorname{Re} b(2m+1-k) + \operatorname{Im} b(k) \operatorname{Im} b(2m+1-k)).$$

The sign of δ, as we see from Eq. (4.23), is given by the imaginary part of the first non-vanishing term $b(n_0)$ in the series (4.21).

5. CALCULATION OF PHASE-SHIFTS FOR NUCLEON-ANTINUCLEON SCATTERING

In Sec. 4 we showed how to deduce the phase-shifts from the scattering matrix for systems of particles of arbitrary spin. As an example we consider nucleon-antinucleon scattering.

We take the S-matrix given by the covariant formulation of perturbation theory,[16]

$$S = \sum_{n=0}^{\infty} S^{(n)}, S^{(n)} = \frac{(-i)^{(n)}}{n!} \int_{-\infty}^{\infty} dx_n \ldots \int_{-\infty}^{\infty} dx_1 P(H(x_n) \ldots H(x_2) H(x_1)) \tag{5.1}$$

The units are chosen so that $h = c = 1$. For pseudoscalar meson theory with pseudoscalar charge-symmetric interaction,

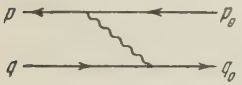
$$H(x) = gN(\bar{\psi}(x)\gamma^5\tau_i\varphi_i(x)\psi(x)). \tag{5.2}$$

Here $\psi(x)$ and $\varphi_i(x)$, $(i = 1, 2, 3)$ are operators of the free nucleon and meson fields, τ_i are nucleon isotopic spin operators, $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$, $(\gamma^5)^+ = -\gamma^5$, and N denotes a normal product. Transition matrix elements can be written down by the usual rules.[4] In the lowest non-vanishing approximation we have

$$S^{(2)}_{1f\leftarrow i} = -\frac{ig^2}{(2\pi)^2} \frac{(\tau_i)_{\mu\mu_0}(\tau_i)_{\nu_0\nu}M^2}{(E_pE_qE_{p_0}E_{q_0})^{1/2}} \frac{\bar{u}_\sigma(\mathrm{p})\gamma^5 u_{\sigma_0}(\mathrm{p}_0)\cdot\bar{v}_{\rho_0}(\mathrm{q}_0)\gamma^5 v_\rho(\mathrm{q})}{(\mathrm{p}-\mathrm{p}_0)^2-(E_p-E_{p_0})^2+\mu^2} \delta(\mathrm{p}+\mathrm{q}-\mathrm{p}_0-\mathrm{q}_0)\delta(E_p+E_q-E_{p_0}-E_{q_0}); \tag{5.3}$$

$$S^{(2)}_{2f\leftarrow i} = \frac{ig^2}{(2\pi)^2} \frac{(\tau_i)_{\mu\nu}(\tau_i)_{\nu_0\mu_0}M^2}{(E_pE_qE_{p_0}E_{q_0})^{1/2}} \frac{\bar{u}_\sigma(\mathrm{p})\gamma^5 v_\rho(\mathrm{q})\cdot\bar{v}_{\rho_0}(\mathrm{q}_0)\gamma^5 u_{\sigma_0}(\mathrm{p}_0)}{(\mathrm{p}_0+\mathrm{q}_0)^2-(E_{p_0}+E_{q_0})^2+\mu^2} \delta(\mathrm{p}+\mathrm{q}-\mathrm{p}_0-\mathrm{q}_0)\delta(E_p+E_q-E_{p_0}-E_{q_0}), \tag{5.4}$$

where $\mathbf{p}_0(\mathbf{q}_0)$ and $\mathbf{p}(\mathbf{q})$ are the initial and final momenta of the nucleon (antinucleon), $\sigma_0(\rho_0)$ and $\sigma(\rho)$ are the spin indices of nucleon (antinucleon) in initial and final states, $\mu_0(\nu_0)$ and $\mu(\nu)$ are the isotopic spin indices of nucleon (antinucleon), M and μ are the nucleon and meson masses, and u and v are bispinors of the form



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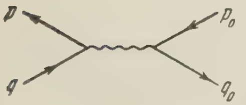

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$$u(\mathrm{p}) = (2M(E_p+M))^{-1/2}\begin{pmatrix} E_p+M \\ \mathrm{p}\sigma \end{pmatrix}; \qquad v(\mathrm{q}) = (2M(E_q+M))^{-1/2}\begin{pmatrix} \sigma\mathrm{q} \\ E_q+M \end{pmatrix}.$$

The Feynman diagrams corresponding to $S_{1f\leftarrow i}$ and $S_{2f\leftarrow i}$ are shown in Figs. a and b.

Passing to the center-of-mass system, we have*

$$\delta(\mathrm{p}+\mathrm{q}-\mathrm{p}_0-\mathrm{q}_0) = \delta(\mathrm{p}+\mathrm{q})\delta(\mathrm{p}_0+\mathrm{q}_0);$$
$$\delta(E_p+E_q-E_{p_0}-E_{q_0}) = \delta(2E_p-2E_{p_0}) = (E_0/2p_0)\delta(p-p_0),$$

Expanding the bilinear combinations of u and v with $\mathbf{q}_0 = -\mathbf{p}_0 = -\mathbf{n}_0 p_0$ and $\mathbf{q} = -\mathbf{p} = -\mathbf{n}p$, we obtain

$$S^{(2)}_{1,2f\leftarrow i} = R_{1,2}\,\delta(\mathrm{p}+\mathrm{q})\delta(\mathrm{p}_0+\mathrm{q}_0)\delta(p-p_0); \tag{5.5}$$

$$R_1 = -\frac{ig^2}{32\pi^2}(\tau_i)_{\mu\mu_0}(\tau_i)_{\nu_0\nu}\frac{p_0}{E_0}\frac{((\mathrm{n}-\mathrm{n}_0)\sigma)_{\sigma\sigma_0}((\mathrm{n}-\mathrm{n}_0)\sigma)_{\rho_0\rho}}{p_0^2(\mathrm{n}-\mathrm{n}_0)^2+\mu^2}; \tag{5.6}$$

$$R_2 = (ig^2/8\pi^2)(\tau_i)_{\mu\nu}(\tau_i)_{\nu_0\mu_0}(E_0/p_0)\delta_{\sigma\rho}\delta_{\sigma_0\rho_0}/(-4E_0^2+\mu^2). \tag{5.7}$$

It is convenient to express these quantities in terms of the outer products $\tau_i \times \tau_k = \tau_{1i}\tau_{2k}$ and $\sigma_i \times \sigma_k = \sigma_{1i}\sigma_{2k}$. This is done by means of

$$(\tau_i)_{\mu\mu_0}(\tau_i)_{\nu_0\nu} = \tau_1\tau_2 - 2\tau_{1y}\tau_{2y}; \tag{5.8}$$

$$(\tau_i)_{\mu\nu}(\tau_i)_{\nu_0\mu_0} = {}^1/_2(3-\tau_1\tau_2+2\tau_{1y}\tau_{2y}); \tag{5.9}$$

$$(\sigma\mathrm{a})_{\sigma\sigma_0}(\sigma\mathrm{a})_{\rho_0\rho} = (\sigma,\mathrm{a})(\sigma_2\mathrm{a}-2\sigma_{2y}a_y); \tag{5.10}$$

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*Hereinafter we write E_0 for E_{p_0}.

$$\delta_{\sigma\rho}\,\delta_{\sigma_0\rho_0} = {}^1/_2\,(1 + \sigma_1\sigma_2 - 2\sigma_{1y}\,\sigma_{2y}). \tag{5.11}$$

The matrices R_1 and R_2 in this representation are formally not invariant* under rotations either in ordinary space or in isotopic space. To obtain R_1 and R_2 in manifestly invariant form, we must transform to a new representation $R^1 = U R U^+$ with $U = \tau_{2y}\,\sigma_{2y}$. Dropping the primes, we obtain in the new representation

$$R_1 = -\frac{ig^2}{32\pi^2}\,(\tau_1\tau_2)\,\frac{p_0}{E_{p_0}}\,\frac{(\sigma_1(\mathrm{n}-\mathrm{n}_0))\,(\sigma_2(\mathrm{n}-\mathrm{n}_0))}{p_0^2(\mathrm{n}-\mathrm{n}_0)^2+\mu^2}\,; \tag{5.12}$$

$$R_2 = \frac{ig^2}{32\pi^2}\,(3+\tau_1\tau_2)\,\frac{E_0}{p_0}\,\frac{1-\sigma_1\,\sigma_2}{(-4E_0^2+\mu^2)}\,. \tag{5.13}$$

The transition to a representation using the total spin $S = {}^1\!/_2\,(\sigma_1 + \sigma_2)$ and $T = {}^1\!/_2\,(\tau_1 + \tau_2)$ is simplified by using the relations

$$(\sigma_1 \mathrm{a})\,(\sigma_2 \mathrm{a}) = 2\,(\mathrm{Sa})^2 - a^2, \quad \sigma_1\sigma_2 = 2S^2 - 3, \;\; \tau_1\tau_2 = 2T^2 - 3. \tag{5.14}$$

The result is then

$$R_1 = \frac{ig^2}{16\pi^2}\,(2T^2-3)\,\frac{p_0}{E_0}\,\frac{1-(\mathrm{nn}_0)-(\mathrm{S}\,(\mathrm{n}-\mathrm{n}_0))^2}{p_0^2\,(\mathrm{n}-\mathrm{n}_0)^2+\mu^2}\,, \quad R_2 = \frac{ig^2}{8\pi^2}\,\frac{T^2\,(2-S^2)}{-4E_0^2+\mu^2}\,\frac{E_0}{p_0}\,. \tag{5.15}$$

Equation (5.15) implies that, for $S = 1$ or for $T = 0$, the annihilation term vanishes in this approximation, and the scattering is given by the exchange term alone.

We consider the case of triplet scattering $(S = 1)$ and expand the matrix R_1 in polynomials $L^{1,1}_{\ell',\ell;\,J}(\mathbf{n},\mathbf{n}_0)$. First we must expand $1/[\mathbf{n}-\mathbf{n}_0)^2 + c^2]$ in Legendre polynomials

$$\frac{p_0}{E_0}\,\frac{1}{p_0^2\,(\mathrm{n}-\mathrm{n}_0)^2+\mu^2} = \sum_{j=0}^{\infty} A_j\,(p_0)\,P_j\,(\mathbf{nn}_0), \tag{5.16}$$

Next, using Eqs. (3.2) — (3.7) of Ref. 1, we obtain

$$\begin{aligned} R_1 = \frac{ig^2}{16\pi^2}\,(2T^2-3)\sum_j \frac{A_j\,(p_0)}{2j+1}\Big\{&-\frac{1}{2j+3}\,L_{11,\,j+1} + \frac{1}{2j+1}\,L_{11,\,j} + \frac{j+2}{2j+3}\,L_{22,\,j+1} - L_{22,\,j} + \frac{j-1}{2j-1}\,L_{22,\,j-1} - \frac{1}{2j+1}\,L_{33,\,j} \\ &+ \frac{1}{2j-1}\,L_{33,\,j-1} - \frac{\sqrt{(j+1)(j+2)}}{2j+3}\,(L_{13,\,j+1} + L_{31,\,j+1}) + 2\,\frac{\sqrt{j\,(j+1)}}{2j+1}\,(L_{13,\,j} + L_{31,\,j}) - \frac{\sqrt{j\,(j-1)}}{2j-1}\,(L_{13,\,j-1} + L_{31,\,j-1})\Big\}. \end{aligned} \tag{5.17}$$

With a change in the summation variable, this expansion takes the form

$$R_1 = \frac{ig^2}{16\pi^2}\,(2T^2-3)\sum_{J=0}^{\infty}\Big\{a_J^{(1)}L_{11,\,J} + a_J^{(2)}L_{22,\,J} + a_J^{(3)}L_{33,\,J} + a_J\,(L_{13,\,J} + L_{13,\,J})\Big\},$$

$$(2J+1)\,a_J^{(1)} = \frac{1}{2J+1}\,A_J\,(p_0) - \frac{1}{2J-1}\,A_{J-1}\,(p_0), \;\; (2J+1)\,a_J^{(2)} = \frac{J}{2J+3}\,A_{J+1}\,(p_0) - A_J\,(p_0) + \frac{J+1}{2J-1}\,A_{J-1}\,(p_0), \tag{5.18}$$

$$(2J+1)\,a_J^{(3)} = \frac{1}{2J+3}\,A_{J+1}\,(p_0) - \frac{1}{2J+1}\,A_J\,(p_0), \;\; (2J+1)\,a_J = \sqrt{J\,(J+1)}\Big(-\frac{1}{2J+3}A_{J+1}\,(p_0) + \frac{2}{2J+1}\,A_J\,(p_0) - \frac{1}{2J-1}A_{J-1}\,(p_0)\Big).$$

In the case of singlet scattering $(S = 0)$ the matrices R_1 and R_2 become

$$R_1 = \frac{ig^2}{16\pi^2}\,(2T^2-3)\,\frac{p_0}{E_0}\,\frac{1-(\mathrm{nn}_0)}{p_0^2\,(\mathrm{n}-\mathrm{n}_0)^2+\mu^2}\,, \quad R_2 = \frac{ig^2}{4\pi^2}\,\frac{T^2}{-4E_0^2+\mu^2}\,\frac{E_0}{p_0}\,. \tag{5.19}$$

Expanding R_1 and R_2 into polynomials $L^{0,0}_{\ell',\ell;\,\ell}(\mathbf{n},\mathbf{n}_0)$ which are simply multiples of Legendre polynomials[1]

$$L^{0,0}_{l',\,l;\,l} \equiv L^0_l\,(\mathbf{n},\,\mathbf{n}_0) = (2l+1)\,P_l\,(\mathbf{nn}_0), \tag{5.20}$$

we find

$$R_1 = \frac{ig^2}{16\pi^2}\,(2T^2-3)\sum_{l=0}^{\infty} b_l\,(p_0)\,L^0_l\,(\mathbf{n},\,\mathbf{n}_0), \tag{5.21}$$

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* This does not mean that there is any failure of spin-conservation, since in this representation $S_x = {}^1\!/_2\,(\sigma_{1x} - \sigma_{2x})$, $S_y = {}^1\!/_2\,(\sigma_{1y} + \sigma_{2y})$, $S_2 = {}^1\!/_2\,(\sigma_{1z} - \sigma_{2z})$, and similarly $T_x = {}^1\!/_2\,(\tau_{1x} - \tau_{2x})$, $T_y = {}^1\!/_2\,(\tau_{1y} + \tau_{2y})$, $T_z = {}^1\!/_2\,(\tau_{1z} - \tau_{2z})$.

$$R_2 = \frac{ig^2}{4\pi^2} \frac{T^2}{-4E_0^2 + \mu^2} \frac{E_0}{p_0} L_0^0 (\mathbf{n}, \mathbf{n}_0), \tag{5.22}$$

where $(2\ell + 1) b_\ell (p_0) = B_\ell (p_0)$ is a coefficient in the expansion of

$$p_0 (1 - (\mathbf{n}\mathbf{n}_0)) / E_0 [p_0^2 (\mathbf{n} - \mathbf{n}_0)^2 + \mu^2]$$

in Legendre polynomials.

We saw in Sec. 4 that the phase shifts are in general to be found by diagonalizing the matrix $a^{s's}_{\ell'\ell, J}$ which is formed from the coefficients in the expansion of the operator $R = S - 1$ in matrix polynomials $L^{s's}_{\ell'\ell, J}$. The elements a_0^k of the diagonalized matrix are related to the phase-shifts δ_k by

$$-2\pi i p_0^2 a_0^k (p_0) = e^{i\delta_k} \sin \delta_k .$$

Consider the case of triplet scattering. When $J = 0$, only one polynomial $L^1_{33, 0}$ is different from zero. Thus, except for

$$a_{33} \equiv a_0^{(3)} = \left(-A_0 (p_0) + \frac{1}{3} A_1 (p_0)\right) \frac{ig^2}{16\pi^2} (2T^2 - 3), \quad A_0 (p_0) = \frac{1}{4p_0E_0} \ln \left(1 + \frac{4p_0^2}{\mu^2}\right), \tag{5.23}$$

$$A_1 (p_0) = \frac{3}{2p_0E_0} \left\{\frac{1}{2}\left(1 + \frac{\mu^2}{2p_0^2}\right) \ln \left(1 + \frac{4p_0^2}{\mu^2}\right) - 1\right\}, \tag{5.24}$$

all elements of the matrix a_{ik} are zero. The single phase-shift, corresponding to the transition ${}^3P_0 \to {}^3P_0$, is given by

$$\sin \delta = (g^2 / 8\pi)\, p_0^2 (2T^2 - 3) (-A_0 (p_0) + {}^1/_3\, A_1 (p_0)). \tag{5.25}$$

Compare Eq. (4.24) of Ref. 1. The phases for transitions with $J \geq 1$ are found similarly.

In conclusion I express my deep gratitude to Academician I. E. Tamm for his valuable criticisms.

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CORRELATION THEORY FOR RAYLEIGH SCATTERING OF LIGHT. I

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A phenomenological theory for Rayleigh scattering of light in an isotropic visco-elastic medium is developed on the basis of the correlation theory proposed earlier by the author for such media. General (i.e., allowing arbitrary dispersion relations for the parameters of the medium) expressions (3.1) are obtained for the spectral intensities of the scattered light corresponding to various polarization conditions. As an application of the general formulas, an expression (4.3) is found for the integrated intensity of the light scattered by isotropic fluctuations (under certain assumptions as to the form of the dispersion relations). The case of a slightly viscous fluid is treated and the appropriate dispersion correction to the Einstein formula is considered.

INTRODUCTION

RECENTLY a large amount of experimental data has been accumulated concerning the Rayleigh scattering of light in liquids; in many cases these data do not agree with the thermodynamic theory of scattering.[1] Substantial discrepancies are found for different characteristics of the polarized component of the scattered light, i.e., the part due to density and temperature fluctuations (so-called isotropic scattering).* Thus, the integrated intensity of the isotropic component of the scattered light is, in many cases, less (sometimes by 20 — 25 percent) than that calculated by the Einstein formula. The ratio of the integrated intensities of the non-displaced line and the Mandel'shtam-Brillouin doublet is considerably greater than that given by the Landau-Placzek formula. Discrepancies between experiment and the thermodynamic theory are especially marked in liquids which exhibit strong dispersion effects in propagation of sound.[1] This is, of course, natural since the existence of frequency-dependent mechanical and thermal medium parameters (and a frequency dependence between these parameters and the index of refraction) makes the thermodynamic theory inapplicable, even in principle. The thermodynamic values of the parameters correspond only to rather low (strictly speaking, infinitely small) frequencies, whereas oscillations with frequencies up to 10^{11} are important in isotropic scattering.

Fabelinskii[1,2] derived a non-thermodynamic formula for the intensity of the doublet, which, because dispersion has been taken into account, is in much better agreement with experiment. This result, together with a detailed analysis of the relevant facts, led Fabelinskii to the conslusion that it is necessary to develop a non-thermodynamic theory of scattering, which will take into account the frequency dependence of the parameters of the scattering medium, The contemporary spectral theory of heat fluctuations can give a complete solution to this problem, which in turn affords one of the most interesting applications of the general theory. The appropriate results are the concern of the present paper.

As is well known, the spectral composition and intensity of light scattered under various polarization conditions is completely determined by the mean-square spectral components of the tensor $\epsilon_{\alpha\beta}$ which describes the fluctuations of the dielectric constant. The spectral amplitudes $\epsilon_{\alpha\beta}$, in turn, are linear functions (because the fluctuations are small) of the quantities which characterize the deviations from complete equilibrium in the medium, i.e., deformation and temperature. Thus the problem reduces to the search for spectral correlation functions for the thermal fluctuations in the deformation and temperature medium. However, the correlation theory for thermal fluctuations in distributed systems which is now available[5] makes it possible to solve this problem in a completely general way; this was done by the author in Ref. 5 for the case of an isotropic (but otherwise arbitrary) medium.* The spectral analysis approach is carried out in the same way in the case of a scattering medium which exhibits dispersion. Before proceeding to an exposition of this analysis it is convenient to characterize briefly some of its basic features.

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*As is well known, thermodynamic theory is, in general, not applicable to scattering by anisotropic fluctuations.

First of all, it is clear that this is a phenomenological theory, i.e., it contains no assumptions as to the dispersion mechanism, and, correspondingly, in no way specifies the dispersion relations, except for the completely general requirement that these relations must not disturb the dissipation properties of the medium. Any further specification of the dispersion relations may be introduced externally.

As noted, the scattering medium is assumed to be isotropic, i.e., characterized by two (complex) elastic moduli, scalar thermal parameters (heat conductivity and of thermal expansion), and scalar mechanical and thermo-optical parameters. Thus, the visco-elastic medium under consideration could be either a liquid of arbitrary viscosity, or an amorphous solid. An analogous theory for crystals would be more complicated only from the computational point of view; in principle it would be formulated in the same way using precisely this same approach.

Since the spectral theory to be described includes arbitrary internal processes in the medium, it comprises in a single scheme both isotropic and anisotropic fluctuations of the dielectric constant, i.e., it describes the entire spectrum of the scattered light, including the so-called wing of the Rayleigh line. By integrating the expressions for the spectral intensities with respect to frequency we can obtain integrated intensities of the total spectrum, and of individual components (e.g., the non-displaced line, the doublet, etc.). In fact, a series of such integrated intensities can be obtained for any reasonable specification of the dispersion relations.

In the first part of the paper, Secs. 1 – 3, we obtain the general formulas (3.1) for the spectral intensities of the scattered light, observed at right angles to the primary wave, for four combinations of the polarization of the primary and scattered light. Then, in Sec. 4, we calculate the integrated intensity of the isotropic scattering I_{iso} under the following assumptions: (1) the mechanical and thermo-optical coefficients

$$Y = -\rho_0 (\partial\varepsilon / \partial\rho)_T, \qquad Z = T_0 (\partial\varepsilon / \partial T)_\rho,$$

and the coefficient of thermal expansion exhibit no dispersion, (2) the compression modulus $\overline{K}$ obeys a dispersion relation with a single relaxation time τ', and the shear modulus $\overline{\mu}$ depends on the frequency in an arbitrary way. In the case of liquids, the Einstein formula follows from Eq. (4.3) for I_{iso}, when Z is neglected and $\overline{K}$ is dispersionless. For slightly viscous fluids (Sec. 5) the dispersion correction to the Einstein formula simplifies considerably. It depends on the ratio of $1/\tau'$ to the frequency Ω_{MB} of the Mandel'stam-Brillouin doublet; however, it does not reduce, as might be expected, to the replacement of the static compressibility β_T in the Einstein formula by the compressibility $1/K(\Omega_{MB})$ at the doublet frequency.

In the second part of the paper, we consider some further consequences of (3.1) under certain assumptions as to the dispersion relations: the separation of I_{iso} into components (central line, Mandelstam-Brillouin doublet, background), the dispersion correction to the Landau-Placzek relation, the frequency behavior of the intensity and the degree of depolarization in the wing of the Rayleigh line, etc.

1. SPECTRAL SOLUTION OF THE SCATTERING PROBLEM

Let us dwell briefly on the well-known spectral solution of the problem of the scattering of a plane monochromatic electromagnetic wave by weak optical irregularities of the medium (cf. Ref. 7), describing this solution in terms of the correlation theory for random fields.

Let the dielectric constant of the medium have the form

$$\varepsilon_0\delta_{\alpha\beta} + \varepsilon_{\alpha\beta}(t, \mathbf{r}),$$

where the fluctuation $\epsilon_{\alpha\beta}$ from the mean value ϵ_0 is small enough so that in calculating the scattered field produced by the incident primary wave

$$\mathbf{E}_0 = \mathbf{A}e^{i(\omega t - \mathbf{k}_0\mathbf{r})}, \quad (k_0 = \omega\sqrt{\varepsilon_0}/c),$$

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* The analogous problem in the case of a viscous fluid with no dispersion was solved earlier by Landau and Lifshitz.[6]

we can confine ourselves to the first perturbation-theory approximation. We seek a solution of Maxwell's equations in spectral form, i.e. we represent $\epsilon_{\alpha\beta}$ in the form

$$\varepsilon_{\alpha\beta}(t, \mathbf{r}) = \int_{-\infty}^{+\infty} \varepsilon_{\alpha\beta}(\Omega, \mathbf{r})\, e^{i\Omega t}\, d\Omega,$$

and the scattered field in the form

$$E_\alpha(t, \mathbf{r}) = \int_{-\infty}^{+\infty} E_\alpha(\Omega, \mathbf{r})\, e^{i(\omega+\Omega)t}\, d\Omega.$$

Then, at distances $R \gg k_0 \ell^2$, where ℓ is a linear dimension of the scattering volume V, the spectral amplitude $E_\alpha(\Omega, \mathbf{r})$ is

$$E_\alpha(\Omega, \mathbf{r}) = \frac{k^2 e^{-ikR}}{4\pi\varepsilon_0 R} \int_V e^{-i\mathbf{q}\mathbf{r}} p_\alpha(\Omega, \mathbf{r})\, d\mathbf{r},$$

where $\mathbf{q} = \mathbf{k}_0 - \mathbf{k}$, $\mathbf{k}$ is the wave vector of the scattered wave $[k = (\omega + \Omega)\sqrt{\epsilon_0}/C \approx k_0$, since only $\Omega \ll \omega$ is important$]$, and $\mathbf{p}$ is the component of the polarization perpendicular to the direction of observation (multiplied by 4π):

$$\mathbf{p} = 4\pi\{\mathbf{P} - \mathbf{k}(\mathbf{k}\,\mathbf{P})/k^2\} \quad P_\alpha = \varepsilon_{\alpha\beta} A_\beta / 4\pi. \tag{1.1}$$

As a measure of the intensity of the scattered light we take $J_{\omega+\Omega}$, the ratio of the average Poynting vectors of the scattered and direct waves, multiplied by $V^{-1}(2\pi\epsilon_0 R/k^2)^2$. An elementary calculation gives

$$J_{\omega+\Omega} = \frac{1}{VA^2} \int_V \int_V \overline{p_\alpha(\Omega, \mathbf{r})\, p_\alpha^*(\Omega, \mathbf{r}')}\, e^{-i\mathbf{q}\rho}\, d\mathbf{r}\, d\mathbf{r}', \qquad \rho = \mathbf{r} - \mathbf{r}'. \tag{1.2}$$

Thus, the intensity at the frequency $\omega + \Omega$ is determined by the spatial autocorrelation of the spectral amplitude of the vector $\mathbf{p}$. Because of the statistical homogeneity of the fluctuation field of $\mathbf{p}$ the correlation function depends on the vectors $\mathbf{r}$ and $\mathbf{r}'$ only through their difference ρ.

Equation (1.2) is limited only by two initial assumptions: the first approximation with respect to $\epsilon_{\alpha\beta}$, and observation in the Fraunhofer zone. Now let (in agreement with actual experimental conditions) the dimensions of the volume V be very large with respect to the correlation distance of the optical inhomogeneities of the medium. The integration over $\mathbf{r}'$ can then be replaced by integration over ρ, with the limits of integration extended to infinity:

$$J_{\omega+\Omega} = \frac{1}{VA^2} \int_V d\mathbf{r} \int_{-\infty}^{+\infty} \overline{p_\alpha(\Omega, \mathbf{r})\, p_\alpha^*(\Omega, \mathbf{r}')}\, e^{-i\mathbf{q}\rho} d\rho = \frac{1}{A^2} \int_{-\infty}^{+\infty} \overline{p_\alpha(\Omega, \mathbf{r})\, p_\alpha^*(\Omega, \mathbf{r}')}\, e^{-i\mathbf{q}\rho} d\rho. \tag{1.3}$$

If we represent $p_\alpha(\Omega, \mathbf{r})$ as a spatial Fourier integral

$$p_\alpha(\Omega, \mathbf{r}) = \int_{-\infty}^{+\infty} p_\alpha(\Omega, \mathbf{q})\, e^{i\mathbf{q}\mathbf{r}} d\mathbf{q}$$

and bear in mind that because of the spatial homogeneity of the fluctuations the correlation function of the $\Omega\mathbf{q}$-amplitude has the form

$$\overline{p_\alpha(\Omega, \mathbf{q})\, p_\alpha^*(\Omega, \mathbf{q}')} = \overline{|p_\alpha(\Omega, \mathbf{q})|^2}\, \delta(\mathbf{q} - \mathbf{q}'),$$

it is apparent that

$$\overline{p_\alpha(\Omega, \mathbf{r})\, p_\alpha^*(\Omega, \mathbf{r}')} = \int_{-\infty}^{+\infty} \overline{|p_\alpha(\Omega, \mathbf{q})|^2}\, e^{i\mathbf{q}\rho} d\mathbf{q}. \tag{1.4}$$

The integral appearing in (1.3) is the inverse of the Fourier integral (1.4), multiplied by $(2\pi)^3$, i.e.,

$$J_{\omega+\Omega} = (2\pi)^3 A^{-2}\, \overline{|p_\alpha(\Omega, \mathbf{q})|^2}. \tag{1.5}$$

Thus, the intensity of the light at frequency $\omega + \Omega$, scattered in the direction $\mathbf{k}$, is to within a constant factor the $\Omega\mathbf{q}$-intensity of the vector $\mathbf{p}$, where $\mathbf{q} = \mathbf{k}_0 - \mathbf{k}$.

Let us now fix the conditions of observation. We take the x-axis to be the direction of propagation of the primary wave and restrict ourselves to the case of perpendicular observation (along the y-axis). Then

$$\mathbf{q} = (k_0, -k_0, 0),\ q^2 = 2k_0^2. \tag{1.6}$$

Moreover, we consider two polarizations of the primary wave, vertical $(A_z = A)$ and horizontal $(A_y = A)$; these will be denoted by the symbols ↑ and →. We also consider two polarizations of the observed scattered light, along z and x. Then, from (1.1), (1.5) and (1.6) we obtain the following expressions for the intensities, corresponding to the four polarization combinations (in what follows we omit the index $\omega + \Omega$ on J):

$$J_x^{\uparrow} = (2\pi)^3 \overline{|\varepsilon_{13}(\Omega, \mathbf{q})|^2},\quad J_z^{\uparrow} = (2\pi)^3 \overline{|\varepsilon_{33}(\Omega, \mathbf{q})|^2},\quad J_x^{\rightarrow} = (2\pi)^3 \overline{|\varepsilon_{12}(\Omega, \mathbf{q})|^2},\quad J_z^{\rightarrow} = (2\pi)^3 \overline{|\varepsilon_{32}(\Omega, \mathbf{q})|^2}. \tag{1.7}$$

Thus, the problem reduces to calculation the Ωq-intensities of the four components of the tensor $\epsilon_{\alpha\beta}$, where the vector **q** is fixed in accordance with (1.6).

2. THE RELATION BETWEEN $\epsilon_{\alpha\beta}$ AND THE FLUCTUATIONS IN DEFORMATION AND TEMPERATURE OF THE MEDIUM

For the mechanical and thermal quantities which give a spectral description of the fluctuations in the scattering medium we take the Ω-amplitudes of the fluctuational deformations $u_{\alpha\beta}$ and the deviation of the temperature T from its equilibrium value T_0. In this connection, it is convenient to divide $u_{\alpha\beta}$ into a pure volume deformation $u'_{\alpha\alpha} \equiv u = -\rho/\rho_0$ (ρ is the deviation of the density from the equilibrium value ρ_0) and a pure shear deformation $u'_{\alpha\beta} = u_{\alpha\beta} - u\delta_{\alpha\beta}/3$. Because the fluctuations are small the relation between $\epsilon_{\alpha\beta}$ and the quantities $u'_{\alpha\beta}$, u, and $\vartheta = T/T_0$ is linear and the spectral-amplitude relations are, by hypothesis, purely algebraic:

$$\varepsilon_{\alpha\beta}(\Omega, \mathbf{r}) = X(i\Omega)\, u'_{\alpha\beta}(\Omega, \mathbf{r}) + Y(i\Omega)\, u(\Omega, \mathbf{r})\, \delta_{\alpha\beta} + Z(i\Omega)\, \vartheta(\Omega, \mathbf{r})\, \delta_{\alpha\beta}. \tag{2.1}$$

The coefficients X, Y, and Z are, in general, complex and describe the mechanical-optical and thermo-optical properties of the medium at the frequency Ω.*

In the case in which these coefficients and the mechanical and thermal parameters of the medium are dispersionless (and consequently are real), Eq. (2.1) is valid even for the quantities which are not resolved spectrally, i.e.,

$$\varepsilon_{\alpha\beta}(t, \mathbf{r}) = Xu'_{\alpha\beta}(t, \mathbf{r}) + Yu(t, \mathbf{r})\,\delta_{\alpha\beta} + Z\vartheta(t, \mathbf{r})\,\delta_{\alpha\beta}.$$

Contracting this expression, we obtain $(u'_{\alpha\alpha} = 0)$

$$\varepsilon_{\alpha\alpha} \equiv 3\varepsilon = 3Yu + 3Z\vartheta,$$

whence it follows that in the absence of dispersion

$$Y = (\partial\varepsilon/\partial u)_\vartheta = -\rho_0(\partial\varepsilon/\partial\rho)_T,\quad Z = (\partial\varepsilon/\partial\vartheta)_u = T_0(\partial\varepsilon/\partial T)_\rho. \tag{2.2}$$

Equation (2.1) is, of course, also valid for the Ωq-amplitudes

$$\varepsilon_{\alpha\beta}(\Omega, \mathbf{q}) = X(i\Omega)\, u'_{\alpha\beta}(\Omega, \mathbf{q}) + Y(i\Omega)\, u(\Omega, \mathbf{q})\,\delta_{\alpha\beta} + Z(i\Omega)\,\vartheta(\Omega, \mathbf{q})\,\delta_{\alpha\beta}. \tag{2.3}$$

It follows from (1.7) and (2.3) that

$$J_x^{\uparrow} = (2\pi)^3 |X|^2 \overline{|u'_{13}|^2},\quad J_z^{\uparrow} = (2\pi)^3 \overline{|Xu'_{33} + Yu + Z\vartheta|^2},\quad J_x^{\rightarrow} = (2\pi)^3 |X|^2 \overline{|u'_{12}|^2},\quad J_z^{\rightarrow} = (2\pi)^3 |X|^2 \overline{|u'_{32}|^2}. \tag{2.4}$$

All the intensities, except $J_z^{\uparrow}$, are due to anisotropic fluctuations; only $J_z^{\uparrow}$ contains the isotropic part, which is proportional to $\overline{|Yu + Z\vartheta|^2}$.

3. SPECTRAL INTENSITIES OF THE SCATTERED LIGHT

In accordance with Eq. (2.4), the determination of the spectral intensities of the scattered light reduces to a search for the Ωq-intensities for deformation and temperature fluctuations. This problem has been solved in Ref. 5 and we now make use of the results obtained there.

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*Since the spectral amplitudes of the fluctuations of any internal parameters can be expressed linearly in terms of $u_{\alpha\beta}(\Omega, \mathbf{r})$ and $\vartheta(\Omega, \mathbf{r})$ (in the absence of spatial dispersion, cf. Ref. 5), there are no grounds in our spectral theory for introducing in (2.1) an isotropic term of the form $\epsilon'\delta_{\alpha\beta}$ which does not reduce to fluctuations of density and temperature, as was done in Ref. 8.

In Secs. 4 and 5 of the work cited formulas are given for the following Ωq-intensities (in the notation of Ref. 5 these are called ωk-intensities):

$$\overline{u'_{\alpha\beta}u'^{*}_{\mu\nu}},\ \overline{u'_{\alpha\beta}u^{*}},\ \overline{u'_{\alpha\beta}\vartheta^{*}},\ \overline{uu^{*}},\ \overline{u\vartheta^{*}},\ \overline{\vartheta\vartheta^{*}}.$$

In these general formulas, in accordance with Eq. (1.6), we set $q_1 = q_2 = q/\sqrt{2}$, and $q_3 = 0$ whence the expressions which appear in (2.4) are obtained. As a result we find

$$J_x^{\rightarrow} = \frac{\Theta |X|^2}{8\pi i\Omega}\left\{\frac{1}{\overline{\mu}^*}\left(1+\frac{d_1\overline{\mu}q^2}{\Delta}\right)-\text{compl. conj.}\right\},\qquad J_z^{\rightarrow} = J_x^{\uparrow} = \frac{\Theta |X|^2}{16\pi i\Omega}\left\{\frac{1}{\overline{\mu}^*}\left(1+\frac{\rho_0^2\Omega^4}{|A_4|^2}\right)-\text{compl. conj.}\right\},$$

$$J_z^{\uparrow} = \frac{\Theta}{2\pi i\Omega}\left\{\frac{|X|^2}{3\overline{\mu}^*}\left(1+\frac{d_1\overline{\mu}q^2}{3\Delta}\right)+\left(XY^*C^*+X^*YC\frac{\overline{\mu}}{\overline{\mu}^*}\right)\frac{C\overline{K}q^2}{3\Delta}-\left(XY^*\frac{\overline{K}}{K^*}+X^*Y\frac{\overline{\mu}}{\overline{\mu}^*}\right)\frac{dq^2}{3\Delta}+\left(XZ^*+X^*Z\frac{\overline{\mu}}{\overline{\mu}^*}\right)\frac{C\overline{K}q^2}{3\Delta}\right. \tag{3.1}$$

$$\left.+\frac{|Y|^2}{\Delta}\left(\frac{d_1A_3}{\overline{K}^*}-CC^*\overline{K}q^2\right)-\left[YZ^*C\overline{K}q^2-Y^*Z\left(C^*A_1-C\frac{\overline{K}}{\overline{K}^*}A_3\right)\right]\frac{1}{\Delta}+\frac{|Z|^2}{\Delta}A_1-\text{compl. conj.}\right\}.$$

For brevity we use the same notation as in Ref. 5, in particular

$$d = D+\varkappa T_0q^2/i\Omega,\ d_1 = d-C^2\overline{K} = D_1+\varkappa T_0q^2/i\Omega,\quad D_1 = D-C^2\overline{K},\ \Delta = dA_1-C^2\overline{K}A_3 = d_1A_3-dKq^2,$$

$$A_1 = \rho_0\Omega^2-\left(\overline{K}+\frac{4}{3}\overline{\mu}\right)q^2,\ A_3 = \rho_0\Omega^2-\frac{4}{3}\overline{\mu}q^2,\ A_4 = \rho_0\Omega^2-\overline{\mu}q^2.$$

Thus, in the final analysis, all quantities are expressed in terms of $\overline{K}$, $\overline{\mu}$, κ, C and D which have the following significance: $\overline{K}$ and $\overline{\mu}$ are the complex elastic moduli, i.e.,

$$\overline{K} = K(\Omega^2)+i\Omega\zeta(\Omega^2);\ \overline{\mu} = \mu(\Omega^2)+i\Omega\eta(\Omega^2),$$

where K is the compression modulus, μ is the shear modulus, ξ and η are respectively the volume and shear viscosity, κ is the complex heat conductivity, and C and D are complex coefficients in the equations which relate the spectral amplitudes of the fluctuations to the specific (per unit mass) entropy S, the volume deformation u and the temperature $\vartheta = T/T_0$:

$$\rho_0T_0S(\Omega,\mathbf{r}) = C\overline{K}u(\Omega,\mathbf{r})+D_1\vartheta(\Omega,\mathbf{r}).$$

In the absence of dispersion $\overline{K}$, $\overline{\mu}$, κ, C, D and D_1 are real constants where

$$K = 1/\beta_T,\ C = \alpha T_0,\ D = \rho_0T_0c_p,\ D_1 = \rho_0T_0c_v, \tag{3.2}$$

where β_T is the isothermal compressibility, α is the coefficient of heat expansion and c_p and c_v are respectively the heat capacities at constant pressure and constant volume.

Equations (3.1) have been written for a frequency which satisfies the condition $\hbar\Omega \ll \Theta = kT_0$. When this condition does not hold Θ must be replaced by the quantum expression for the mean energy of an oscillator.

Without carrying out a more detailed analysis of Eq. (3.1) we can make several general remarks as to the form of the spectra which are described.

As has been noted in Ref. 5 the dispersion equation can be broken up into two equations, namely $\Delta = 0$ and $A_4 = 0$ where the first refers to the compression wave and the thermal wave while the second refers to the shear wave. Correspondingly, the terms in Eq. (3.1) which contain Δ in the denominator give rise to a triplet in the spectrum—a central maximum and a Mandel'shtam-Brillouin doublet while those containing A_4 in the denominator (that term appears only in $J_z^{\rightarrow} = J_x^{\uparrow}$) give rise to a shear doublet.

The presence of dispersion in the medium leads, first of all, to a finite width in the maxima of both doublets. In the absence of loss these maxima become discrete lines (δ-peaks); this also occurs at the central maximum if the heat conductivity approaches zero. Secondly, dispersion leads to the appearance of a wide continuous background, the distribution of which depends on the dispersion relation. Terms which contain $\overline{\mu}$ in the denominator and which arise solely as a result of optical anisotropy (for $X = 0$ these all disappear) lead to the formation of the wing of the Rayleigh line which, in the case of a liquid, may extend far beyond the Mandel'shtam-Brillouin doublet. In a similar way terms with $\overline{K}$ in the denominator make a contribution to the continuous background. These terms will be considered in greater detail later on.

It is reasonable to assume that the ratio of the maxima to the background level itself depends, in an important way, on the values of the parameters. Thus, for example, the triplet in $J_X^{\rightarrow}$, which should be present because of the second term in the denominator which contains Δ, is practically unobservable in a slightly viscous fluid because μ is very small at the frequencies in question. If the value of the static shear modulus approaches zero (transition to the liquid case) the maxima of the shear doublet, described by the second term in $J_Z^{\rightarrow} = J_X^{\uparrow}$, contract about the central line and merge into the background, and so on.

Equations (3.1) represent the most information that can be reasonably expected from a purely phenomenological theory. These general expressions allow us to draw concrete conclusions which are necessari based on certain assumptions as to the dispersion relations between the parameters X, Y, Z and $\bar{K}$, $\bar{\mu}$, C, D. There are certain theoretical restrictions on the enumerated parameters. The dispersion of the mechanical and thermal parameters of the medium, $\bar{K}$, $\bar{\mu}$, C and D may be subject to rather general considerations—something like those which were introduced in the formulation of the relaxation theory for dispersion and absorption of sound in liquids in the work of Mandel' shtam and Leontovich.[9,10] The problem is somewhat more difficult for the heat conductivity κ. As for the parameters X, Y, Z there seems to be, as yet, no approach to a theory for setting up a frequency dependence for these parameters. However, Eqs. (3.1) make it possible to compare various assumptions with the experimental data on the scattering of light and can be useful in this respect.

We now turn to the results which follow from Eq. (3.1) when some particular assumptions as to the dispersion relations between the above-mentioned parameters are introduced.

4. INTEGRATED INTENSITY OF THE ISOTROPIC COMPONENT OF THE SCATTERED LIGHT

The intensity of the isotropic component, given by those terms in Eqs. (3.1) which do not contain X, is

$$J_{z\,\mathrm{iso}}^{\uparrow} = \frac{\Theta}{2\pi i\Omega}\left\{\frac{1}{\Delta}\left[|Y|^2\left(\frac{d_1 A_3}{\bar{K}^*} - CC^*\bar{K}q^2\right) - YZ^*C\bar{K}q^2 + Y^*Z\left(C^*A_1 - C\frac{\bar{K}}{\bar{K}^*}A_3\right) + |Z|^2 A_1\right] - \text{compl. conj.}\right\}. \tag{4.1}$$

The indices which indicate the vertical polarization of the primary and observed waves will be discarded in the following.

To compute the integrated intensity

$$I_{\mathrm{iso}} = \int_{-\infty}^{+\infty} J_{\mathrm{iso}}\,d\Omega \tag{4.2}$$

we make use of (7.4) add (7.5) of Ref. 5, having made the following assumptions for this purpose:

(a) Y, Z and C are independent of Ω, that is, they have real thermodynamic values (2.2) and (3.2).

(b) The compression modulus $\bar{K}$ obeys a simply dispersion relation characterized by a single relaxation time τ'

$$\bar{K} = \frac{K_\infty z + K_0/\tau'}{z + 1/\tau'} = K_\infty\frac{z + a}{z + 1/\tau'}\ \left(z = i\Omega,\ \ a = \frac{K_0}{K_\infty \tau'}\right).$$

Then,

$$J_{\mathrm{iso}} = \frac{\Theta}{2\pi i\Omega}\left\{\frac{1}{\Delta}\left[- Y^2C^2\bar{K}q^2 + YZC\left(A_1 - \bar{K}q^2\right) + Z^2A_1 + \left(Y^2d_1 - YZC\right)\frac{A_3(z - 1/\tau')}{K_\infty(z - a)}\right] - \text{compl. conj.}\right\}.$$

Substituting this relation in (4.2) and using (7.5) of Ref. 5 in the first three terms (and their complex conjugates) and (7.4) in the last term (and its complex conjugate) we obtain

$$I_{\mathrm{iso}} = \Theta\left\{Y^2\left(\frac{1}{K_\infty} - \frac{4\mu_0/3}{K_0\left(K_0 + \frac{4}{3}\mu_0\right)}\right) + \frac{Z^2}{D_{1\infty}} + \frac{2\left(K_\infty - K_0\right)}{K_0K_\infty\Delta_a}Y\left(Yd_{1a} - ZCK_a\right)\right\}. \tag{4.3}$$

The subscript 0 and ∞ here denote values of the parameters which obtain at $z \to 0$ and $z \to \infty$; these can be shown to be real from general considerations. The subscript a denotes values for $z = a = K_0/K_\infty\tau'$ (also real).

For a liquid ($\mu_0 = 0$), neglecting the dependence of ϵ on temperature ($Z = 0$), in the absence of dispersion in the compression modulus ($K_\infty = K_0 = 1/\beta_T$) the Einstein thermodynamic formula[11] is obtained from Eq. (4.3)

$$I_{\text{iso}} = \Theta Y^2/K_0 = \Theta \beta_T [\rho_0 (\partial \varepsilon/\partial \rho)_T]^2.$$

Equation (4.3) describes both a highly viscous liquid and a solid body, depending on the degree to which the assumptions made above apply. In any case, in Eq. (4.3) we have not assumed that the dispersion differential $K_\infty - K_0$ is small; for the shear modulus $\bar{\mu}$ there is, in general, no limitation other than the general requirement that the limiting values μ_0 and μ_∞ be finite. As an example we now consider the case of a slightly viscous liquid.*

Let us consider first the assumption that the coefficient Y is constant. The measurements of Fabelinskii and Motulevich have shown[12] in a number of liquids that even at supersonic frequencies (10^7) the square of the modulus of $Y(i\Omega)$ is about 10 — 12 percent lower than at 50 cps; this result is in good agreement with the values obtained from the intensity of the scattered light. It is difficult to interpret this result even when we compare the frequency at which the liquid parameters change Ω, not with the frequency of light ω, but with the reciprocal values of the time required for establishing the index of refraction. This time should be determined, at least in terms of order of magnitude, by the width of the absorption band and not by its location in the spectrum. Thus, for example, the index of refraction of a gas through which passes a wave front with an associated frequency of approximately 10^{15} is established not in 10^{-15} sec but approximately 10^{-9} sec, corresponding to the width of the spectral line of the gas ($\Delta\lambda \sim 0.01$ A). If the line (or band) has a width of 1 — 10 A the corresponding time is $10^{-11} - 10^{-12}$ sec. Thus one expects a certain dispersion effect in $Y(i\Omega)$ for frequencies of the order of 10^{10} but under the condition that the light frequency is close to that of the absorption line or band. This is precisely what is not found in scattering observations and thus the index of refraction should follow the density oscillations almost quasi-statically if the medium parameters very in a sinusoidal manner.

Thus, it is difficult to explain dispersion in $Y(i\Omega)$ even at frequencies of the order of 10^{10}, let alone 10^7. In view of these considerations the assumption that Y is constant is a reasonable one; however, the contradictory experimental results force us to leave this assumption open to question.

5. CASE OF SLIGHTLY VISCOUS LIQUID

Turning now to the case of a liquid ($\mu_0 = 0$) with rather small dispersion we need keep in (4.3) only terms of first order in the dispersion corrections, that is, in the coefficient for $K_\infty - K_0$ we take values of all parameters for $z \to 0$.† Furthermore, since the value of Z is itself small, in general it is possible to neglect the dispersion corrections in terms which contain Z. Then Eq. (4.3) assumes the form

$$I_{\text{iso}} = \Theta \left\{ Y^2 \left(\frac{1}{K_\infty} + \frac{2(K_\infty - K_0)s}{K_0^2(1+s)} \right) + \frac{Z^2}{D_{10}} \right\}, \tag{5.1}$$

where

$$s = -\frac{d_{1a}A_{3a}}{d_a K_a q^2} \approx -\frac{d_{10}A_{30}}{d_0 K_0 q^2} = \frac{1+\gamma\chi q^2\tau'}{1+\chi q^2\tau'} \frac{1}{\Omega_{MB}^2 \tau'^2} \approx \frac{1}{\Omega_{MB}^2 \tau'^2}, \tag{5.2}$$

$K = \kappa_0 T_0/D_0 = \kappa_0/\rho_0 c_p$ is the heat conductivity of the liquid, $\gamma = D_0/D_{10} = c_p/c_v$ and $\Omega_{MB}^2 = \gamma K_0 q^2/\rho_0$ (Ω_{MB} is the frequency of the Mandel'shtam-Brillouin doublet). It is apparent that the parameter s is a measure of the degree to which the compression modulus relaxes at the doublet frequency. If the relaxation region for K is already reached when $\Omega = \Omega_{MB}$, ($\Omega_{MB}\tau' \gg 1$), s is small and the corresponding dispersion correction in (5.1) is unimportant.

In the three liquids for which Fabelinskii and his colleagues[13,14] either measured or estimated the values of τ', if it is assumed that $q^2 = 4.15 \times 10^{10}$ (light wave length 4368 A) we have:

Liquid	$\tau' \cdot 10^{10}$	γ	$\chi q^2 \tau'$	$\Omega^2_{MB} \cdot 10^{-20}$	s
Carbon tetrachloride	0.67	1.44	0.0021	4.31	0.52
Benzol	2.44	1.44	0.0092	7.89	0.021
Carbon bisulfide	21.8	1.56	0.114	5.59	0.00038

Thus the dispersion correction given by the second term in (5.1) is important in carbon tetrachloride but is extremely small in the other two liquids.

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*In solid bodies information concerning the dispersion of elastic moduli, or even the order of magnitude of the relaxation time is not available, to the best of the author's knowledge.

†It is obvious that the quasi-static values coincide with the thermodynamic values (dispersionless); hence, for these, as for the parameters which were earlier assumed to be dispersionless, we will frequently use the notation of (3.2).

In those cases in which $s \ll 1$ Eq. (5.1) reduces to the following:

$$I_{\text{iso}} = \Theta\left(\frac{Y^2}{K_\infty} + \frac{Z^2}{D_{10}}\right) = \Theta\beta_T\left\{\rho_0^2\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T^2\frac{K_0}{K_\infty} + \frac{\gamma-1}{\alpha^2}\left(\frac{\partial\varepsilon}{\partial T}\right)_\rho^2\right\}. \tag{5.3}$$

If K_∞ could be measured independently, using Eq. (5.3) it would be possible to verify the assumption that the coefficient Y is dispersionless. But dispersion measurements in the propagation of compression waves do not give direct information as to the magnitude of K_∞.

Actually, it follows from Eq. (6.5) of Ref. 5 that the square of the complex velocity of propagation of longitudinal waves v is given as follows:

$$v^2 = \frac{P+R}{2\rho_0 D_1}, \quad P = i\Omega\varkappa\rho_0 T_0 + D\overline{K} + \frac{4}{3}D_1\overline{\mu}, \quad R = \sqrt{P^2 - 4i\Omega\varkappa\rho_0 T_0 D_1(\overline{K} + 4\overline{\mu}/3)}. \tag{5.4}$$

The parameters D, D_1, and K reach their limiting values D_∞, $D_{1\infty}$ and K_∞ at frequencies $\Omega \gg 1/\tau'$, that is, when $\Omega \approx 10^{11}$. But in the case of a slightly viscous fluid these frequencies are very small as compared with $1/\tau$ (τ is the relaxation time for the shear modulus), since $\tau'/\tau \sim 10^3$. Furthermore, at the frequencies indicated the quantity $\Omega\kappa\rho_0 T_0$ is very small as compared with $D_0 K_0$ since these quantities can become comparable (i.e., comparable thermal and longitudinal wave length) only when

$$\Omega = D_0 K_0 / \varkappa\rho_0 T_0 = 1/\rho_0\beta_T\chi \sim 10^{13}.$$

Thus, in Eq. (5.4) we can neglect $4\overline{\mu}/3$ and $\Omega\kappa\rho_0 T_0$ not only when $\Omega \to 0$ but also for those frequencies $\Omega \gg 1/\tau'$ at which the velocity has already reached value v_∞. Then

$$v^2 = D\overline{K}/D_1\rho_0, \tag{5.5}$$

Whence it follows that from the dispersion differential in the velocity v we can find the limiting value ($\Omega\tau' \gg 1$) of the "adiabatic" modulus $K_{\text{ad}} = D\overline{K}/D_1$

$$\frac{v_0^2}{v_\infty^2} = \frac{D_0 K_0}{D_{10}} : \frac{D_\infty K_\infty}{D_{1\infty}} = \frac{1}{\beta_S}\frac{D_{1\infty}}{D_\infty K_\infty} \qquad \left(\beta_S \doteq \frac{\beta_T}{\gamma}\right), \tag{5.6}$$

whereas in (5.3) we have the limiting value K_∞ of the "isothermal" modulus $\overline{K}$.

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ON K_{e3} DECAY

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The electron and π meson energy distributions from K_{e3} decay are calculated. Measurement of these distributions will make it possible to establish the type of decay interaction and to determine the strong interaction form factors $g(E_\pi)$ in these decays.

1. The investigation of K_{e3} decays ($K^\pm \to e^\pm + \nu + \pi^0$ and $K^0 \to e^\pm + \nu + \pi^\pm$) is very important for the clarification of the character of weak electron interactions. In the general case, any matrix element for the K_{e3} decay of a K meson at rest, which does not contain products of lepton functions, is of the form

$$\mathfrak{M} = \{g_S \bar{\psi}_e \psi_\nu + g_V \bar{\psi}_e \gamma_4 \psi_\nu + i g_T \bar{\psi}_e \gamma_4 \gamma \psi_\nu \mathbf{k}_\pi M^{-1}\} (2M^{3/2} E_\pi^{1/2})^{-1}. \tag{1}$$

Here $g_{S,V,T}$ are functions of the π meson energy E_π corresponding to the scalar (S), vector (V), and tensor (T) interactions. The dependence of g on E_π cannot be calculated, since there exists no theory for strong interaction between the K and π mesons. The matrix element $\mathfrak{M}$ is normalized to make the $g(E_\pi)$ functions dimensionless and constant in first-order perturbation theory with respect to strong interaction between the K and π mesons. Further, M is the mass of the K meson, $k_\pi = E_\pi^2 - m_\pi^2$, and $\hbar = c = 1$. (For a detailed discussion of the form of $\mathfrak{M}$, see the works of Furuichi et al.[1] and Pais and Treiman.[2]

2. With the aid of (1) one easily obtains an expression for the probability of emitting an electron with energy E_e and a π meson with energy E_π, namely

$$W(E_\pi, E_e)\, dE_\pi dE_e = \{|g_S|^2 [(M - E_\pi)^2 - k_\pi^2] + |g_V|^2 [k_\pi^2 - (M - E_\pi, - 2E_e)^2] \tag{2}$$

$$+ |g_T|^2 [(M - E_\pi)^2 - k_\pi^2] [M - E_\pi - 2E_e]^2 M^{-2} + i(g_S g_T^* + g_S^* g_T) [(M - E_\pi)^2 - k_\pi^2] [M - E_\pi - 2E_e] M^{-1}\} (32\pi^3 M^3)^{-1} dE_\pi dE_e.$$

In considering the electron spectrum at a fixed π meson energy, it is convenient to write (2) in the form

$$W(\varepsilon) = \Phi_S + \Phi_V [\varepsilon_0^2 - (1 - \varepsilon)^2] + \Phi_T (1 - \varepsilon)^2 + \Phi_{ST} (1 - \varepsilon). \tag{3}$$

Here the $\Phi_{S,V,T,ST}$ depend only on E_π, and are independent of the E_e, and

$$\varepsilon = 2E_e / (M - E_\pi), \quad \varepsilon_0 = k_\pi / (M - E_\pi), \quad 1 - \varepsilon_0 \leqslant \varepsilon \leqslant 1 + \varepsilon_0.$$

Equation (3) is equivalent to Eq. (8) of Pais and Treiman.[2] However, the choice of the electron energy E_e as the variable (rather than the angle between the electron and π meson) makes Eq. (3) clearer. It follows from (2) that $\Phi_{S,V,T} > 0$, with the sign and magnitude of Φ_{ST} being determined by the relative phases of g_S and g_T. If time (combined) parity is conserved, all the g are real.* If $\Phi_{ST} = 0$, it is seen from (3) that $W(\epsilon)$ is symmetric about $\epsilon = 1$. Lack of such symmetry would indicate the presence both of the S and the T interactions. As is also seen from (3), the presence of a maximum at $\epsilon = 1$ in the spectrum would indicate the presence of the V interaction, whereas a minimum would indicate the T interaction. If it were to turn out that the experimental data is not consistent with (3), this would indicate that the weak lepton interaction is nonlocal.

3. A measurement of the electron spectrum for fixed E_π that would give complete information on the type of interaction is, however, a difficult experimental problem. In this connection it is of interest to obtain expressions for the electron and π meson spectra $W(E_e)dE_e$ and $W(E_\pi)dE_\pi$, which are obtained by integrating Eq. (2) over E_π and E_e, respectively. The integration over E_π can be performed

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*We note that the function g_T in Eq. (1) differs by a factor of i from f_T of Pais and Treiman.[2] Therefore their assertion that invariance under time reversal corresponds to real $f_{S,V,T}$ is incorrect.

The author is grateful to B. L. Ioffe and I. M. Smushkevich for discussing this question.

only under certain assumptions as to the form of $g(E_\pi)$. Furuichi et al.[1] have performed this integration and obtained an expression for $W(E_e)dE_e$ on the assumption that $g(E_\pi) = \text{const}$. Matinian[3] has also obtained an expression for $W(E_e)dE_e$ for the S interaction. Comparing their formulas with the experimental data, the authors[1] conclude that $g_T \neq 0$.

The integration over E_e, which can be performed without any assumptions as to the form of $g(E_\pi)$, gives

$$W(E_\pi)\,dE_\pi = \{|g_S|^2(M^2+m_\pi^2-2ME_\pi)k_\pi + |g_V|^2 2k_\pi^3/3$$
$$+ |g_T|^2(M^2+m_\pi^2-2ME_\pi)k_\pi^3/3M^2\}\,dE_\pi/32\pi^3M^3, \quad m_\pi \leqslant E_\pi \leqslant (M^2+m_\pi^2)/2M = E_{\pi\,\mathrm{max}}. \tag{4}$$

(For the S interaction, Matinian[3] has previously an expression for $W(E_\pi)dE_\pi$.) From Eq. (4) it follows in particular, that by measuring the π meson spectrum near its upper limit one can establish the presence or absence of the V interaction even without knowing the form of $g_{S,V,T}(E_\pi)$. Indeed, for the S and T interactions, $W(E_{\pi\,\mathrm{max}}) = 0$ in all cases, whereas for the V interaction $W(E_\pi\,\mathrm{max}) = 0$ only if $g_V = 0$ (it is unlikely that $g_V(E_\pi)$ vanishes at this point accidentally).

The author is grateful to Professor I. Ia. Pomeranchuk and Academician L. D. Landau for constant interest in the work, and to Academician A. I. Alikhanov for discussing the possible experimental measurements of the K_{e3} decay spectra.

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LETTERS TO THE EDITOR

THE EFFECTIVE CROSS SECTION OF THE Be^9(n, 2n) REACTION

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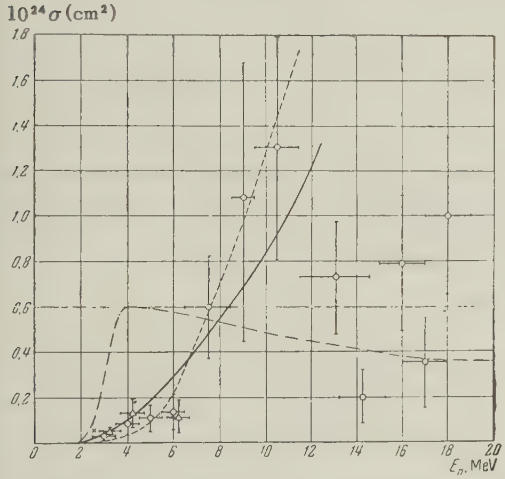
We have investigated the course of the cross section for the Be^9(n, 2n) reaction and obtained certain values of the cross sections of the principally competing reactions emitting charged particles — Be^9(n, α) He^6 and Be^9(n, t) Li^7 — under the action of neutrons from the reaction Li(d, n) with energies from 1.5 to 19 Mev.

To reduce the number of scattered neutrons, the irradiation of the lithium target was carried out, after extraction from the cyclotron, with a focused beam of 4-Mev deuterons. The beam of fast neutrons was carefully collimated with a system of slits.

In the case of the Be^9(n, 2n) reaction two α-particles are emitted, the tracks of which can be observed, in the form of two-prong stars, in photoemulsions loaded with Be^9. To facilitate the observation of the stars formed in reactions with Be^9, a specially prepared disk was prepared which contained Be^9 powder with grain size of ~ 1—2μ between two emulsion layers. More than 200 stars from Be^9 grains were found which were related to the reaction Be^9(n, 2n) and the reactions Be^9(n, α) He^6 and Be^9(n, t) Li^7.



Effective cross section of the Be^9(n, 2n) reaction: ○ — obtained in this work; dashed curve — from the formula of Sachs[4] for R = 4.44, a = —1.16; dash-dot curve — from the formula of Sachs for R = 1.39, a = 0.463; solid curve — from Mamasakhlisov.[1] The effective cross section of the Be^9(n, α) He^6 reaction; Δ — our work; × — from Allen.[6] The effective cross section of the Be^9 (n, t) Li^7 reaction; □ — our work.

Assuming a known direction for the incident neutron, and with the help of the laws of conservation of energy and momentum, the energy of the primary neutron E_p was calculated for each star. In the case of the Be^9(n, 2n) reaction the total energy of the two emitted neutrons and the polar angle were calculated from the sum of the momenta relative to the direction of the primary neutron. The competing reactions with emission of charged particles were identified by calculation. The neutron spectrum for each series of irradiations was determined from the proton recoil spectrum observed in the emulsion pellicles used for the study of the reactions. The values of the effective cross sections of the Be^9(n, 2n) reaction and of the principally competing reactions Be^9(n, α) He^6 and Be^9(n, t) Li^7, as a function of the energy of the neutrons, is shown on the figure. The principal errors for each cross section point are composed of the errors in the measurement of the neutron beam (≈ 12%), errors in the determination of the amount of material in the emulsion pellicle (≈ 15%), and statistical errors.

As can be seen from the figure, the dependence of the cross section obtained by us in the interval 4 to 10 Mev coincides with the cross section dependence calculated by Mamasakhlisov.[1] Besides, the average effective cross section for incident neutrons with energies from 2 to 11 Mev, $\overline{\sigma}$ = 200 ± 120 mb, found by us in these experiments, agrees within the limits of error with the results on the average effective cross section for a Ra + Be source: $\overline{\sigma}$ = 300 ± 100 mb.[2,3]

Recently Sachs[4] has suggested a semi-empirical model, describing a direct decay into light nuclei under the action of low-energy nucleons, and examined as an example the $Be^9(n, 2n)\,Be^8$ reaction. Using the value of $\bar{\sigma}$ (see Refs. 2 and 3) for the spectrum of a Ra + Be source, he obtained a curve for the effective cross section of the $Be^9(n, 2n)\,Be^8$ reaction which differed sharply from Mamasakhlisov's curve and from our experimental results (see figure). A set of constants, characterizing the size of the interaction, derived from our data would yield $R = 4.44$ and $a = -1.16$, and using these the dependence of the cross section would change and become similar to Mamasakhlisov's curve. In this way, the dependence of the cross section of the $Be^9(n, 2n)$ reaction obtained experimentally agrees, within the limits of error, with the dependence of the cross section given by Ref. 1 and by Sachs' formula with corrected coefficients in the energy interval from 4 to 10 Mev. It is not possible, on the basis of the obtained experimental results, to make a choice between the assumptions made in these papers on the reaction mechanism, but it is possible that it will allow in the future the working over of the available experimental material.

As can be seen from the drawing, the theoretically determined dependence of the cross section from the quoted works differs from the experimental dependence for energies greater than 10 Mev. There is no reason to assume the existence of any appreciable errors for high energies. Because of this one may think that for high energies another reaction mechanism begins to operate to an appreciable extent. Indeed the following types of reactions are possible:

$$(1)\ \ Be^9(n, n')\,Be^{9*} \rightarrow n'' + Be^8(2\alpha); \qquad (2)\ \ Be^9(n, 2n)\,Be^8(2\alpha); \qquad (3)\ \ Be^9(n, 2n2\alpha).$$

Calculation of the excitation energy for Be^8 at $E_n \leq 10$ Mev has indicated the presence of known levels of this nucleus, which is evidence of the formation of Be^8 in the course of the $Be^9(n, 2n)$ reaction, that is of the prevalence of reactions (1) and (2) for this primary neutron energy.

For energies greater than 10 Mev, the third type of reaction apparently plays a part to an appreciable extent: the simultaneous decay of the compound nucleus into two neutrons and two α-particles, which follows from the comparison of the α-particle energy distribution, obtained from experiment, with the distribution computed on the principle given in the work of Ref. 5. At the same time, in the center-of-mass system, an anisotropy is observed with a preferential forward emission of the neutrons and α-particles. This provides a basis for assuming also the existence of a direct interaction of the incident neutron with one of the subgroups of the Be^9 nucleus.

The effective cross section, obtained by us for the $Be^9(n, \alpha)\,He^6$ reaction (see figure) for corresponding values of the neutron energy, agrees within the limits of error with the results obtained in other works.[6]

Besides several disintegration events were found, which can be related to the $Be^9(n, t)\,Li^7$ reaction, and for which the obtained cross sections are exhibited on the same drawing.

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MODULATION EFFECTS IN NUCLEAR MAGNETIC RESONANCE

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MODULATION effects in nuclear magnetic resonance were experimentally observed and explained in a previous work by the author[1] in terms of the dynamics of nuclear magnetization. The existence of such effects has been noted also in optical and microwave spectroscopy.[2] In nuclear magnetic resonance, the relaxation processes in the most important substances (diamagnetic liquids) take place very slowly, and the instantaneous values of the nuclear magnetization M are observed in the form of large changes of the signal waveform even for very slow variation of the external field. In order to choose the experimental conditions and to interpret the results, in this case, it is necessary to use the theory in the form of the solution of the Bloch equations[3] or other more general ones. The solutions for weak rf fields[4-6] and for low modulation[7] are entirely valid, but their use restricts the choice of experimental conditions. In particular, they do not allow one to use conditions in which the greatest signal strength may be attained. Mathematically, this corresponds to the fact that the solutions contain small parameters in the form of factors. The present note attempts to show that for large relaxation times[3] T_1 and T_2, one may obtain strong signals in a form which is easily interpreted.

Let us consider the Bloch equations for a sinusoidally modulated field $H_Z = H_0 + H_m \cos \Omega t$, namely

$$dF/d\tau + [\beta + i(\delta + \varkappa \cos\tau)]F + \eta M_z = 0, \quad dM_z/d\tau + \alpha M_z - \eta v = \alpha M_0;$$
$$\tau = \Omega t, \quad F = v + iu, \quad \alpha = 1/\Omega T_1, \quad \beta = 1/\Omega T_0, \quad \delta = (|\gamma| H_0 - \omega)/\Omega, \quad \eta = |\gamma| H_1/\Omega, \quad \varkappa = |\gamma| H_m/\Omega, \tag{1}$$

where $2H_1$ is the amplitude and $\omega/2\pi$ is the frequency of the rf magnetic field, and γ is the gyromagnetic ratio of the nuclei. We shall attempt to find a solution by treating the parameters α, β, η, which determine the components of the modulation structure, as quantities of the same order of smallness.[1-5] We shall consider the quantity $\delta + k$ a small parameter of the same order, where k is an integer. This means that we shall treat the neighborhood of one of the points in which the solution has a resonant character. We shall consider the modulation amplitude arbitrary, since from the point of view of the signal strength the optimum condition corresponds to[5] $\kappa \sim 1$. Let us introduce the arbitrary parameter ϵ as a factor into those terms of Eq. (1) containing the small parameters. We then have

$$dF/d\tau + i(\varkappa\cos\tau - k)F + \varepsilon[\beta + i(\delta + k)]F + \varepsilon\eta M_z = 0, \qquad dM_z/d\tau = \varepsilon\alpha M_z - \varepsilon\eta v = \varepsilon\alpha M_0 \tag{2}$$

and wish to find a steady-state periodic solution in the form of a power series in ϵ, that is $F = F_0 + \epsilon F_1 + \ldots$, and $M_Z = M_{Z0} + \epsilon M_{Z1} + \ldots$

We obtain sets of differential equations for the coefficients in these series, the first two of each set being

$$dF_0/d\tau + i(\varkappa\cos\tau - k)F_0 = 0, \quad dM_{z0}/d\tau = 0; \tag{3a}$$

$$dF_1/d\tau + i(\varkappa\cos\tau - k)F_1 = -[\beta + i(\delta + k)]F_0 - \eta M_{z0}, \quad dM_{z1}/d\tau = \eta v_0 + \alpha(M_0 - M_{z0}). \tag{3b}$$

Then setting $\epsilon = 1$, we obtain a solution of (1) in the form of the series $F = F_0 + F_1 + \ldots$, and $M_Z = M_{Z0} + M_{Z1} \ldots$, where the index on the term gives its order of smallness.

It is sufficient for our purposes to find the zeroth approximation, i.e.,

$$F_0 = A\exp\{i(k\tau - \varkappa\sin\tau)\}, \quad M_{z0} = \text{const}, \tag{4}$$

where A and M_{Z0} are constants which are determined by the condition that time dependent terms are absent. For Eqs. (3b) we have

$$A = -M_0 \frac{J_k\eta[\beta - i(\delta + k)]}{\beta^2 + (\delta + k)^2 + (\beta/\alpha)J_k^2\eta^2}, \quad M_{z0} = M_0 \frac{\beta^2 + (\delta + k)^2}{\beta^2 + (\delta + k)^2 + (\beta/\alpha)J_k^2\eta^2}.$$

Here J_k is the k-th order Bessel function of the argument κ. Comparison with experiment is more convenient in terms of the Fourier series

$$F_0 = -M_0\eta \sum_{l=-\infty}^{\infty} J_k J_l \frac{e^{i(k-l)\tau}[\beta - i(\delta + k)]}{\beta^2 + (\delta + k)^2 + (\beta/\alpha)J_k^2\eta^2}.$$

The solution obtained has an extremely simple geometric meaning. The vector M precesses about the direction of H_Z. The time variation of M_Z and the vertex angle of the cone of precession are both small, the fundamental effect being the variation of the phase of precession as a result of the variation of H_Z. For weak rf fields, Eq. (4) agrees with the previous solution.[5] The general structure of the solution hardly changes in the interval $1 \gg \eta \gtrsim \beta$. Similarly as in the case of "slow transmission"[3] the effect of saturation is the weakening and broadening of the signal components. The displacement of the side components towards the central one [1,8] is an effect of order η and does not occur in the zeroth approximation. The amplitudes of the dispersion components (corresponding to terms containing the factor $\delta + k$) increase monotonically with the rf field, and the amplitudes of the absorption components attain their maximum value at $\eta = (\alpha\beta)^{1/2}/J_k$, that is when $|\gamma|\ H_1 = (T_1 T_2)^{-1/2}\ J_k$, and then decrease. The maximum values of the amplitudes are independent of the small parameters. Thus, for instance, the maximum amplitude of the central component of the dispersion signal (the term proportional to sin τ) is $2M_0J_1\,(T_2/T_1)^{1/2}$. It is significant that the conditions for the optimum signal strength are attained within the region of applicability of Eq. (4). The condition that Eq. (4) be applicable corresponds to choosing a frequency high enough for the nuclear magnetic resonance signal to have a resolved modulation structure.

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MULTIPLE PAIR PRODUCTION IN QUANTUM ELECTRODYNAMICS

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THE results of recent studies* of the production of electron-positron pairs in photographic emulsions by cosmic rays apparently confirm the reality of the previously observed[1,2] cases of production of two electron-positron pairs in a single act.

It has been pointed out repeatedly that effects of multiple pair production can be used as a criterion for the applicability of the present quantum theory of electromagnetic interactions, just in the range of phenomena in which the correspondence principle can no longer be used. Since up to this time there have not been in the literature any statements on the theory of multiple production of electron-positron pairs, it seemed worth while to make an estimate of the values of the effective cross-sections in question. In the present note we present approximate expressions for the effective cross-sections for production of two electron-positron pairs in the collision of an electron with an atomic nucleus and in the collision of a photon with a nucleus. These processes correspond to the 5th and 6th approximations of the perturbation theory in quantum electrodynamics, so that the direct computation of the values of these cross-sections involves extremely great labor.

The estimates of the cross-sections given below were obtained by the use of a considerably simpler method, the usefulness of which in the high-energy region that concerns us here was considered in a paper by Oppenheimer.[3] The effective cross-section for the production of two pairs in an electron-nucleus collision, $e^- + Z \rightarrow 2(e^- + e^+) + e^- + Z$, can thus be expressed in terms of the effective cross-section for the emission of two Bremsstrahlung γ-ray quanta of frequencies ω', ω'' (ω', $\omega'' \gg m$) and the asymptotic values of the pair-conversion coefficients for γ-ray quanta, $\beta(\omega)$.

Using the formula of Gupta[4] for the Bremsstrahlung radiation of two γ-ray quanta

$$d\sigma \approx \frac{28}{3\pi} r_0^2 \alpha^2 Z^2 \frac{d\omega'}{\omega'} \frac{d\omega''}{\omega''},$$

where $\alpha = 1/137$ and $r_0 = e^2/m$, we have for the total cross-section for the process of formation of two pairs by an electron hitting a nucleus

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*I express my gratitude to V. A. Tumanian, who communicated to me the results obtained by a group in the laboratory of I. I. Gurevich.

$$\sigma_{2e} = \int d\sigma(\omega', \omega'')\,\beta(\omega')\,\beta(\omega''),\ \ \beta(\omega) \approx \frac{2\alpha}{3\pi} \ln \frac{\omega}{m}\,. \tag{1}$$

Apart from an undetermined numerical coefficient of the order of unity, we get finally for the total cross-section for the effect of production of two electron-positron pairs by an electron hitting a nucleus in the extreme relativistic approximation (with logarithmic accuracy):

$$\sigma_{2e} \approx (r_0^2 Z^2 \alpha^4/\pi^3) \ln^4 (E_0/m). \tag{2}$$

(Here and in what follows E_0 is the energy of the initial particle, electron or photon.) We note that in principle the formula (2) could be obtained in the framework of the Weitzsäcker-Williams method[5,6] by using the effective cross-section for the production of two electron-poitron pairs by a γ-ray quantum in the field of the nucleus: $\gamma + Z \rightarrow 2(e^- + e^+) + Z$. Therefore for the effective cross-section for the formation of two pairs by a photon hitting the nucleus we must have

$$\sigma_{2\gamma} \approx (r_0^2 Z^2 \alpha^3/\pi^2) \ln^2 (E_0/m). \tag{3}$$

Comparing Eq. (2) with the Landau-Lifshitz formula[7] for the production of one pair by an electron hitting a nucleus

$$\sigma_{1e} \approx (r_0^2 Z^2 \alpha^2/\pi) \ln^3 (E_0/m)^3,$$

and also comparing Eq. (3) with the Bethe-Heitler formula[8] for the production of one pair by a photon colliding with a nucleus

$$\sigma_{1\gamma} \cong r_0^2 Z^2 \alpha \ln (E_0/m),$$

we come to the conclusion that the expansion parameter in terms of the n of pairs produced is

$$\varepsilon = \sigma_{n+1}/\sigma_n \leqslant (\alpha/\pi)^2 \ln (E_0/m).$$

The formulas given here, Eqs. (2), (3), do not include the effect of screening of the nuclear field by the atomic electrons. As is well known, this latter effect to a considerable degree hinders the increase of the cross-sections in the high-energy region. Therefore it is not excluded that effects of greater interest might be those of production of pairs in collisions with electrons, the effective cross-sections for which can be obtained from Eqs. (2) and (3) by simply replacing Z^2 by unity. It must be remarked, however, that according to Ref. 4 the effect of screening on the effects considered above is relatively small.

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RADIATION INSTABILITY IN NUCLEAR MAGNETIC RESONANCE EXPERIMENTS

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In experiments on nuclear magnetic resonance, the substance being investigated is placed in an rf coil which is part of a resonant circuit. Analysis of the oscillating system comprising the specimen with precessing nuclear magnitization and the resonant circuit shows that in certain cases one cannot neglect the change caused in the rf magnetic field within the winding by the coupling of the specimen to the circuit. It has been shown[1-3] that this change in the field leads to increased damping of the free precession of the nuclear magnetization and to broadening of the signals observed under steady-state conditions.

The present note considers the stability of the system consisting of the specimen and the resonant circuit. The properties of the sample are characterized by the macroscopic Bloch equations.[4] It is assumed that the radiofrequency field acting on the sample is due only to the emf induced in the circuit by the sample, i.e., that the circuit is not connected to an external signal generator. The circuit is tuned close to the free-precession frequency of the nuclear moments, and the transmission band of the circuit is assumed large compared with the width of the nuclear magnetic resonance line. Under these assumptions we have

$$\begin{gathered}\dot{M}_x - \gamma H_0 M_y + M_x/T_2 = 0,\ \dot{M}_y + \gamma H_0 M_x - \gamma H_x M_z + M_y/T_2 = 0,\\ \dot{M}_z + \gamma H_x M_y + M_z/T_1 = M_0/T_1,\ H_x = -(4\pi\eta Q/|\gamma|H_0)\,\dot{M}_x.\end{gathered} \tag{1}$$

Here η is a space factor almost equal to unity when the coil is completely filled by the substance being investigated, and Q is the figure of merit of the circuit.

Let us assume that at time $t = 0$, we have $M_x = M_y = H_x = 0$, and $M_z \neq M_0$. Then one has the obvious solution

$$M_x = M_y = H_x = 0,\ M_{z0} = M_0 + A\exp(-t/T_1), \tag{2}$$

where A is a constant. To test for stability, let us consider the set of equations for small deviations from (2), i.e.,

$$\dot{M}_x - \gamma H_0 M_y + M_x/T_2 = 0,\ \dot{M}_y + \gamma H_0 M_x - \gamma H_x M_{z0} + M_y/T_2 = 0,\quad \dot{M}'_z + M'_z/T_1 = 0,\ H_x = -(4\pi\eta Q/|\gamma|H_0)\,\dot{M}_x, \tag{3}$$

where $M'_z = M_z - M_{z0}$. In Equation (3) terms of higher order have been dropped. The time dependence of M_x is given by

$$\ddot{M}_x + (4\pi M_{z0}|\gamma|\eta Q + 2/T_2)\,\dot{M}_x + (\gamma^2 H_0^2 + T_2^{-2})M_x = 0. \tag{4}$$

It can be shown by separating out the oscillating factor in the solution of this equation, that small deviations from $M_x = M_y = H_x = 0$ increase exponentially if

$$-2\pi M_{z0}|\gamma|\eta Q > 1/T_2. \tag{5}$$

Contrary to the statements of Bloembergen and Pound,[2] the state in which the nuclear magnetization is oriented against the field is not nonradiative when condition (5) is fulfilled. Instead of a purely relaxational change in M_z, self-excitation will occur with the nuclear magnetization approaching its equilibrium value more rapidly and part of the magnetic energy being converted into the energy of high-frequency vibrations. Numerical calculations similar to those performed by Bloembergen and Pound[2] show that condition (5) for self-excitation can be fulfilled, for instance, in experiments with water at attainable values of H_z.

The process of self-excitation when $2\pi|M||\gamma|\eta Q \gg 1/T_2$ is described by the equation

$$d\vartheta/dt = -2\pi|M||\gamma|\eta Q\sin\vartheta,\quad \vartheta = \arccos(M_z/|M|).$$

The solution $\tan(\vartheta/2) = \exp(-2\pi|M||\gamma|\eta Qt)$ of this equation has been used to describe radiation damping.[2] When $t < 0$ this same solution can be used to study the process of self-excitation which is accompanied by the appearance of an inductive signal proportional to $\operatorname{sech}(2\pi|M||\gamma|\eta Qt)$ in the circuit.

The system comprising the sample and the resonant circuit can be used, in the self-excitation state and close to it, as a nuclear magnetic signal generator and amplifier. Fluctuations of the nuclear magnetization are low, so that one could expect to obtain noise factors close to unity in this way. Continuously variable apparatus of this type could be designed with continuously replaceable operating substances.

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ELECTRON CAPTURE CROSS SECTION

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As has been shown by experiment,[2] Gluckstern's[1] calculations of the electron capture cross section σ_3 for singly charged ions of light elements give values which are too high and which decrease too slowly as the ion velocity v is increased. The calculated values are proportional to $v^{-3.5}$, whereas the experimental values for nitrogen ions in nitrogen and argon are proportional to v^{-5}. The calculations are based on classical concepts, which are applicable only when $\kappa = 2iv_0/v \gg 1$, where i is the charge of the ion, and $v_0 = e^2/\hbar$.

For $i \sim 3$ and v/v_0 lying between 3 and 6, when κ is close to unity the method used by Gluckstern becomes inapplicable. In this case σ_3 can be calculated by the statistical method used by Bohr[3] in evaluating the electron capture cross section for fast α particles. The cross section is given in the form of the product $\sigma_1 fn$, where σ_1 is the cross section for the collision process between the ion and electron, in which the electron gains an energy of order $mv^2/2$ (here m is the electron mass), f is the electron capture probability after such a collision, and n is the number of electrons per atom of the substance which participate effectively in the capture. For ions which have a large part of their electron shells, σ_1 is expressed by the same formula, namely $\sigma_1 \sim 4\pi i^2 e^4/(mv^2)^2$, as for α particles only in the case in which the collision diameter $b = 2ie^2/mv^2$ is large enough for the internal structure of the ion to have no effect in its collision with the electron. If b is less than the dimensions of the ion, then σ_1 will be greater due to the increased effective charge. Calculation shows that for that fraction of the ions for which i lies between 0.3 Z and 0.6 Z (here Z is the nuclear charge of the ion) and whose velocities lie between $1.5iv_0/Z^{1/3}$ and $2Z^{2/3}v_0$, the cross section is given by

$$\sigma_1 \sim 4\pi a_0^2 i Z^{1/3} v_0^3 / \sqrt{2}\, v^3,$$

where $a_0 = \hbar^2/me^2$. The maximum binding energy of an electron after capture is given by the ionization potential I of the ion with charge i—1. For $v > u = (2I/m)^{1/2}$, the capture probability f is of the order of $(u/v)^3$. When ions pass through heavy gases, $n \sim Z_2^{1/3} v/v_0$, where Z_2 is the nuclear charge of the atoms in the medium. For ions passing through hydrogen at a velocity not much greater than v_0, we have $n \sim 1$. For nitrogen ions whose charge lies between 2 and 4, for which $u^3 = 0.8 i^2 v_0^3$, the electron capture cross section in argon and nitrogen in the velocity range between $2v_0$ and $5v_0$ is given by the formula

$$\sigma_3 = q \cdot 4\pi a_0^2 i^3 Z_2^{1/3} (v_0/v)^5,$$

where q is a quantity of order unity which takes account of the approximate nature of the calculation. For nitrogen ions passing through hydrogen,

$$\sigma = q \cdot 4\pi a_0^2 i^3 (v_0/v)^6.$$

The dependence of σ_3 on v, i, and Z_2 obtained here is in agreement with the experimental data.[2]

The absolute values of the calculated cross sections lie closest to the experimental ones with $q \approx 2/3$. At lower velocities one should expect the v dependence of σ_3 to be weaker. For $v < u$ and $v < v_0 i/Z^{1/3}$, when $f \sim 1$ and $\sigma_1 \sim 4\pi a_0^2 i^2 (v_0/v)^4$, the cross section σ_3 in heavy gases should be proportional to v^{-3}. The expression for σ_3 in this case agrees with the formula of Bohr and Lindhard[4] up to a constant factor. At high velocities this formula gives values of σ_3 close to those obtained by Gluckstern.

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ABSORPTION OF SECOND SOUND IN ROTATING HELIUM II

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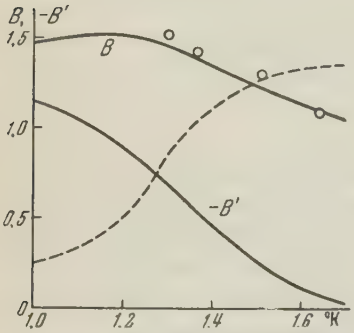
In a very interesting work Hall and Vinen[1,2] have carried out an experimental investigation of the absorption of second-sound oscillations in rotating helium II, and have also given a qualitative theory for this effect, based on the ideas of Onsager and Feynman concerning the excitation of vortices in helium II when it is rotated.*

A calculation of the "mutual friction" force between the superfluid and normal-fluid components requires first of all an investigation of the interaction of the elementary excitations (rotons, at temperatures which are not excessively low) with the vortex filaments. Besides this it is necessary to take account of hydrodynamic effects not connected with specific properties of the elementary excitations (entrainment of the normal fluid by a vortex filament, and motion of a vortex relative to the superfluid). While Hall and Vinen's investigation of the latter does not give rise to any doubts at the moment, their calculation of the scattering of rotons by vortices does not seem to be correct. Their calculation was carried out in the Born approximation, whereas for the process in question the conditions are actually closer to the opposite limiting case — the quasiclassical approximation. In view of this we are inclined to regard the satisfactory qualitative agreement with experiment which Hall and Vinen obtained as to a certain extent fortuitous (it should also be noted that an undetermined empirical constant figured in their calculation).



We have computed the scattering of rotons by vortices in the quasiclassical approximation. The following result is obtained for the force acting per unit length of filament:†

$$\mathbf{f} = D(\mathbf{v}_R - \mathbf{v}_L) + D'[\boldsymbol{\omega}, \mathbf{v}_R - \mathbf{v}_L]/\omega, \quad D \approx 1.2\varkappa\rho_n\sqrt{\mu kT}/p_0, \quad D' = \varkappa\rho_n \tag{1}$$

(where the symbols have the same meaning as in Refs. 1 and 2). The circulation about the vortex is taken to be $\kappa = 2\pi\hbar/m$; other (multiple) values seem unlikely from a theoretical point of view. In comparison with (1), Hall and Vinen's value of D is about 10/T times larger, and their $D' = 0$ by virtue of the properties of the Born approximation.

Using (1) and taking account of the above-mentioned hydrodynamic effects, as did Hall and Vinen,[2] we calculate the total mutual friction force F_{sn} as a function of $v_s - v_n$, and obtain for their coefficients B and B′ [Eq. (11) of Ref. 2].

$$B = \frac{2\rho}{\rho_n\rho_s\varkappa}\left(\frac{1}{E} + \frac{D}{D^2 + D'^2}\right)\Big/\left[\left(\frac{1}{E} + \frac{D}{D^2 + D'^2}\right)^2 + \left(\frac{D'}{D^2 + D'^2} - \frac{1}{\varkappa\rho_s}\right)^2\right]; \tag{2}$$

$$\frac{B'}{B} = \left(\frac{1}{\varkappa\rho_s} - \frac{D'}{D^2 + D'^2}\right)\Big/\left(\frac{1}{E} + \frac{D}{D^2 + D'^2}\right) \tag{3}$$

[for $D' = 0$ these expressions reduce to Eqs. (12) and (13) of Ref. 2].

Understandably we can make a comparison with experimental data only for a temperature region in which it is possible to speak of a "gas" of rotons, and their mean free path is small compared with the distances between vortex lines. Values of B calculated from Eqs. (1) and (2) are shown in the figure (dotted line); we see that at the higher temperatures they agree well with Hall and Vinen's measurements, but at 1.3°K they are already much too small.

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*We take advantage of the opportunity to observe that the "laminar" model of rotating helium II investigated in Ref. 3 is thermodynamically less favorable than the "vortex" model, and therefore it ought to be discarded.

†A calculation shows that the contribution of the phonons to the friction force becomes important only at temperatures on the order of 0.5°K.

This circumstance apparently points to a more complicated character for the interaction of rotons with vortices which cannot be reduced to simple scattering (in a $\mathbf{p}\cdot\mathbf{v}_s$ field). Apparently, for rotons passing by a vortex at small distances from its axis, processes having the character of "strong interactions" take place, which bring about a momentum transfer of the order of the total momentum of the roton. Several processes can take part in the creation of this interaction, such as the transition of a roton to a "bound" state and subsequently its emission back to a free state (for $\mathbf{p}\cdot\mathbf{v}_s < 0$ the field in which the roton finds itself looks like a field of attraction, and a finite motion in it is possible with an energy $\epsilon < 0$; a roton can go over into such an orbit with the simultaneous excitation of vortex oscillations, and can leave by absorbing the energy of these oscillations.* The "eroding" of a vortex due to its natural oscillations can also play an important role. An investigation of all these effects, however, is very involved, and to a certain extent becomes indeterminate because all of the characteristic lengths turn out to be comparable with atomic distances.

It is natural to try to describe this interaction phenomenologically by introducing a temperature-independent effective vortex diameter ("width") a, corresponding to the transfer of the total momentum of a roton to a vortex. The result is the addition of a term $0.7\,a\rho_n\sqrt{kT/\mu}$ in D. This process makes no contribution to D′; on the contrary one has to introduce a factor $\varphi(a) < 1$ in D′ to account for the corresponding "cut-off" of the integral over the range of distances in the calculation of this quantity. Satisfactory agreement with Hall and Vinen's measurements (shown by circles in the figure) is obtained for $a \cong 10$ A (with $\varphi(a) = 0.6$).† This value is somewhat larger than one would naturally expect. It should be emphasized, however, that it is extremely sensitive to the choice of the experimental values of B. In connection with this situation it would seem to be very desirable to carry out further measurements, particularly at lower temperatures.

Values of B′ calculated from Eq. (3) (for the indicated value of a) are also shown in the figure; it should be pointed out that they are very sensitive with repect to the value of the factor $\varphi(a)$. It would also be desirable to obtain experimental data for this quantity (there are none at present).

We express our thanks to Academician L. D. Landau for a discussion of the problems under investigation.

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[3] L. D. Landau and E. M. Lifshitz, Dokl. Akad. Nauk S.S.S.R. 100, 669 (1955)

Translated by W. M. Whitney
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ON THE CONSERVATION OF COMBINED PARITY ‡

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It has been shown by experiments of Wu and others that in the weak interactions (of the type of β-decay) the parity I is not conserved. It has also been shown that in the weak interactions there are two possibilities: either I, C, and T are separately not conserved, and only their product is conserved, or there is conservation of the combined parity IC introduced by Landau[1] and of T. On the other hand, it is believed that in strong and electromagnetic interactions I, C, and T are separately conserved.

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*It can be shown that this process ought not to lead to the appearance of a large longitudinal friction force (along ω). The point is that the longitudinal component of the momentum lost by the roton is transerred to quanta of oscillations of the vortex filaments, belonging to the normal fluid, and consequently transfers of momentum from the normal fluid to the superfluid do not take place.

†The value of this factor depends essentially on the method of "cut-off," that is, on the location of the segment a relative to the axis of the vortex. Ths value of φ given above corresponds to placing this segment off to one side of the axis, where the attaction of a roton takes place.

‡Presented at the Seminar of the Laboratory of Theoretical Physics of the Joint Institute for Nuclear Research, April 20, 1957.

We propose and examine the following hypothesis: In the construction of the Lagrangians of the strong, electromagnetic, and weak interactions of elementary particles it is necessary to require invariance with respect to IC and T, i.e., the conservation of only the combined parity. In some cases the conservation of the combined parity leads to the conservation of spatial parity; in other cases spatial parity is not conserved. The Lagrangian of an interaction invariant with respect to the operation IC can be written in the form of two terms, $L = L_1 + L_0$, where L_0 is invariant with respect to spatial reflection and L_1 is responsible for nonconservation of the parity I.

We confine ourselves to the consideration of the strong and electromagnetic interactions, regarding the connection between spin and statistics as already established. The renormalized and gauge invariant Lagrangian of the interaction of the nucleon field with the electromagnetic field and with fields of particles of spin zero, invariant with respect to the operation IC, has the following form

$$L(x) = e\overline{\psi}_p\gamma_\mu\psi_pA_\mu + g_1(\overline{\psi}_p\gamma_5\psi_n\Phi^* + \overline{\psi}_n\gamma_5\psi_p\Phi^*) + g_2'\overline{\psi}_p\gamma_5\psi_p\Phi^0 + g_2''\overline{\psi}_n\gamma_5\psi_n\Phi^0$$
$$+ ig_3(\overline{\psi}_p\psi_n\Phi - \overline{\psi}_n\psi_p\Phi^*) + g_4(\overline{\psi}_p\psi_n\varphi + \overline{\psi}_n\psi_p\varphi^*) + g_5'\overline{\psi}_p\psi_p\varphi^0 + g_5''\overline{\psi}_n\psi_n\varphi^0 + ig_6(\overline{\psi}_p\gamma_5\psi_n\varphi - \overline{\psi}_n\gamma_5\psi_p\varphi^*). \qquad (1)$$

Here there are also two kinds of bosons, transforming under the operation C in the following way:

$$\varphi^{0\prime}(x) = \varphi^0(-\mathbf{x}, x_4),\ \varphi'(x) = \varphi^*(-\mathbf{x}, x_4),\quad \Phi'^0(x) = -\Phi^0(-\mathbf{x}, x_4),\ \Phi'(x) = -\Phi^*(-\mathbf{x}, x_4). \qquad (2)$$

Besides the vertex coefficients e, g_1, g_2', g_2'', g_4, g_5', g_5'' which exist in the case in which I and C are separately conserved, in our case there have appeared the additional coefficients g_3 and g_6. We note that all the coupling constants $e, g_1, \ldots, g_6$ are real, and the matrices γ are those of Feynman's convention.

In the case of quantum electrodynamics, because of the condition of gauge invariance, the requirement of invariance with respect to the operation IC leads automatically, without the neglect of any terms of the interaction Lagrangian, to invariance with respect to I and C separately, i.e., to the conservation of spatial parity. In the case of meson theory, however, there appear terms leading to nonconservation of I, which gives the possibility in principle of an experimental check of our proposed hypothesis.

Thus while Lee and Yang[2] regard processes in which parity is not conserved as an exception to a general rule, caused by the participation of two-component neutrinos in such processes, we start from a general law of the conservation of the combined parity, and in the special case of the electromagnetic interaction there is at the same time conservation of the spatial parity.

Let us look into the question of experimental checks of our hypothesis. As was shown by Landau,[1] in the case of conservation of the combined parity particles cannot have dipole moments. It is shown above that in the case of electrodynamics the requirement of invariance with respect to IC automatically leads to conservation of I and C separately. Therefore the experimental data from the field of atomic spectroscopy cannot contradict our hypothesis. Thus of the experiments considered by Lee and Yang,[3] which point to the conservation of spatial parity in the strong interactions, there remain nuclear spectroscopy and experiments to measure in double scattering an asymmetry with respect to the plane of the first scattering when protons interact with nuclei of spin zero. The transition amplitude for such a process contains a term that violates the conservation of spatial parity, which leads to the appearance in the second scattering of an up-down asymmetry with respect to the plane of the first scattering. The contribution of the terms responsible for the violation of parity is small; Chamberlain and others[4] have shown that this asymmetry is not larger than 1%. Therefore it is necessary to carry out more precise experiments to measure the above-mentioned asymmetry to test the hypothesis here proposed.

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Translated by W. H. Furry

MESON PRODUCTION IN COLLISIONS OF HIGH-ENERGY NUCLEONS

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IT was shown in the report of D. I. Blokhintsev[1] at the CERN symposium that at energies of several Bev it is possible to distinguish between central and periphery collisions of nucleons and to evaluate the latter by the Weizsäcker-Williams method (see Ref. 2). In this note a more detailed consideration of the periphery collisions of nucleons is given, making more precise the results of Blokhintsev.

Firstly, terms from mesons of different charges are calculated. It follows from the hypothesis of charge independence that virtual $\pi^\pm$ and π^0-mesons are in the nucleon cloud with relative probabilities of $\frac{2}{3}$ and $\frac{1}{3}$. Therefore the energy carried by the π^0-meson cloud of a fast nucleon is

$$E^0 = \frac{2\pi}{3}\int\limits_r^\infty b\,db \int\limits_{-\infty}^{\infty} v\,[\nabla_x\varphi(b,x,t)]^2_{x=0}\,dt,$$

where r is the radius of the nucleon core,[1] and the integrand is the current density of the energy of the meson field of the nucleon, moving with velocity v along the x-axis, at a point on the plane x = 0.

$$\varphi(b,x,t) = -\frac{\gamma g}{\sqrt{4\pi}}\left(\frac{m}{2M}\right)(\sigma\nabla)\frac{\exp\left[-R^{-1}\sqrt{b^2+\gamma^2(x-vt)^2}\right]}{\sqrt{b^2+\gamma^2(x-vt)^2}}$$

is the function of the meson field of the fast nucleon in a non-relativistic approximation for the nucleon; m and M are the masses of the π-meson and nucleon,

$$R = \hbar/mc,\ \gamma = (1-\beta^2)^{-1/2},\ \beta = v/c.$$

Expanding $\nabla_x\varphi(b,x,t)$ at $x = 0$ in a Fourier integral,[2] we obtain

$$E^0 = E\frac{m}{M}g^2\int\limits_0^\infty z\,Q^0(r,z)\,dz,$$

where

$$Q^0(r,z) = \frac{1}{3}\frac{z}{2\pi}\{2uK_0(u)K_1(u) + r^2[K_0^2(u) - K_1^2(u)]\},\quad u = \left(\frac{r}{R}\right)\sqrt{1+\left(\frac{M}{m}z\right)^2}$$

is treated as a spectrum of virtual π^0-mesons accompanying the fast nucleon. Here $K_0(u)$, $K_1(u)$ are the modified Bessel functions, $z = \epsilon/\beta E$, where E is the total energy of the nucleon and ϵ is the energy of the virtual π-meson.* The spectra of $\pi^\pm$-mesons accompanying the proton and neutron, respectively, are equal to $Q^+ = Q^- = 2Q^0$.

The effective cross sections for production of mesons and the relative energy losses of the fast nucleon in the interaction of its cloud with the core of the stationary nucleon [(π, K)-interaction] are equal, respectively, to[2]

$$\sigma_1 = g^2\frac{m}{M}I_1;\ I_1 = \frac{1}{\beta}\int\limits_0^{z_{\max}}[\sigma^\pm(\beta Ez)\,Q^\pm(r,z) + \sigma^0(\beta Ez)\,Q^0(r,z)]\,dz,$$

$$\delta_1 = \frac{\Delta E}{E} = \frac{I^2}{I_1};\quad I_2 = \int\limits_0^{z_{\max}}[\sigma^\pm(\beta Ez)\,Q^\pm(r,z) + \sigma^0(\beta Ez)\,Q^0(r,z)]\,z\,dz,$$

where $\sigma^{+,-,0}(\epsilon)$ with different indices are the total cross sections for scattering of the corresponding π-meson by the nucleon (proton or neutron) where, corresponding to the hypothesis of charge independence, $\sigma^0(\epsilon) = \frac{1}{2}[\sigma^+(\epsilon) + \sigma^-(\epsilon)]$ and $\sigma^+(\epsilon)$ and $\sigma^-(\epsilon)$ are taken from experiment.[4] The use of the experimental cross section for $\sigma_{\pi K}(\epsilon)$ is possible only if the (π,π)-interaction of the nucleon clouds with each

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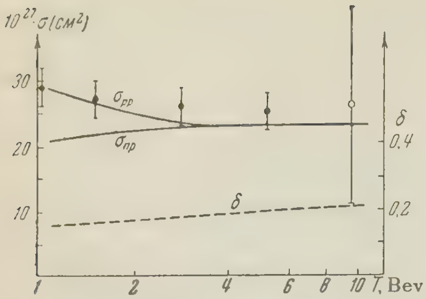
*The cut-off of the meson spectrum $z_{\max} = \epsilon_{\max}/\beta E$, where $\epsilon_{\max} = E - \sqrt{(E + Mc^2)(Mc^2/2)}$ corresponds to the loss by the fast nucleon losing all its kinetic energy in the center-of-mass system of the colliding nucleons.



Relative energy losses δ and effective cross sections for meson production in the periphery collisions of nucleons: σ_{pp} — proton with proton; σ_{np} — neutron with proton. Experimental points are taken from Ref. 6.

other[1] is assumed small which, it appears, is valid for nucleons of energy $E \leq 10$ Bev, since the interaction cross section of such nucleons $\sigma_{NN} \approx 30 \times 10^{-27}\,\mathrm{cm}^2 < \pi(2R)^2$.

Secondly, the interaction of the core of the fast nucleon with the cloud of the stationary one[3] by (K, π) – interaction was take into account. In the rest system of the fast nucleon this interaction occurs as (π, K) – interaction with cross section σ_1. Here the relative energy losses of the fast nucleon, expressed in the laboratory system, are made up of the losses $\delta_2^{\pi} \approx \gamma(1-\beta)\delta_1 \approx \delta_1/2\gamma$ in the production of π-mesons and $\delta_2^{N} \approx \delta_1^2/2\gamma(1-\delta_1)$ in nucleon recoil.

Since the meson field functions employed correspond to the exchange of one π-meson by the interacting nucleons, the cross sections for production of π-mesons in both processes are added together, and the relative energy losses are averaged

$$\sigma = 2\sigma_1; \quad \delta = (\delta_1/2)\,[1 + 1/2\gamma(1-\delta_1)].$$

The best agreement with experiment is obtained for a radius of the nuclear core $r = \hbar/m_K c \approx 2\hbar/Mc$ and a coupling constant $g^2 = 15$ (see figure). Here the cross section for central collision of the nucleons (with impact parameter $(b \leq r)$ $\delta_{KK} \approx \pi r^2 = 5.6 \times 10^{27}\,\mathrm{cm}^2$ constitutes about 20% of the total cross section for inelastic collisions, which coincides with the experimental estimate of Smorodin and others.[5] The relative energy losses of the fast nucleon in the periphery collisions also agrees with experiment.[5] It is essential that they are completely independent of the magnitude of the coupling constant.

In conclusion I should like to express deep gratitude to Prof. D. I. Blokhintsev for valuable advice and interest in this work.

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REMARKS ON ELASTIC SCATTERING OF RELATIVISTIC PARTICLES IN MATTER IN THE STEADY STATE

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LET us consider a steady stream of some type α of positive-rest mass particles scattering in a substance by means only of elastic collisions. These particles are unstable and can be absorbed by the substance. Ordinarily the velocities of the particles comprising the substance are much less than those of the particles of type α. Let us therefore assume that the particles of matter are at rest. Under these conditions, in the rest system of the matter, the kinetic equation of motion of the particle flux in the substance[1] can be written in a form which is identical to the nonrelativistic kinetic equation, namely

$$\boldsymbol{\nu}\nabla\Psi + (\partial/\partial\boldsymbol{\nu})(\boldsymbol{\eta}\Psi) + \nu\Psi/l = \int \Psi^{*}\nu^{*} \sum_{i} \rho_i h_i(\nu^{*}, \nu)\,\delta(\cos\theta_0 - \mu_i)\, d\boldsymbol{\nu}^{*} + q; \quad \Psi^{*} = \Psi(\mathbf{r}, \boldsymbol{\nu}^{*}),\ \boldsymbol{\nu} = (2E/p^2)\,\mathbf{p}, \tag{1}$$

where E is the kinetic energy, and p is the momentum of the particles of type α. During the intervals between collisions, the motion of the particles of type α is given by the equations

$$\frac{dx}{\nu_1} = \frac{dy}{\nu_2} = \frac{dz}{\nu_3} = \frac{d\nu_1}{\eta_1(\mathbf{r}, \nu)} = \frac{d\nu_2}{\eta_2(\mathbf{r}, \nu)} = \frac{d\nu_3}{\eta_3(\mathbf{r}, \nu)} = \frac{d\tau}{1 - \nu^2/4c^2}; \tag{2}$$

where τ is the proper time for this particle, and c is the velocity of light. For Eq. (1), only the first five of Eqs. (2) are relevant.

In any region D of the variables r and ν, the number of particles of type α is

$$\int_D \Psi(\mathbf{r}, \boldsymbol{\nu})(1 + \nu^2/4c^2)\, d\mathbf{r}\, d\boldsymbol{\nu}, \tag{3}$$

while the number of such particles created in this region per unit time is

$$\int_D q(\mathbf{r}, \boldsymbol{\nu})\, d\mathbf{r}\, d\boldsymbol{\nu}; \tag{4}$$

where $\rho_i = \rho_i(\mathbf{r})$ is the density of particles of type i in the substance,

$$1/l = m/pT_0 + \sum_i \rho_i \sigma_i(\nu);$$

$\sigma_i(\nu)$ is the total interaction cross section between particles of type α and those of type i, $1/T_0$ is the probability for spontaneous decay of the particles of type α per unit proper time, θ_0 is the angle between ν^* and ν,

$$\mu_i = \frac{m + m_i}{2m}\frac{\nu}{\nu^{*}} + \frac{m - m_i}{2m}\frac{\nu^{*}}{\nu}; \tag{5}$$

and m_i is the rest mass of a particle of type i. The function $h_i(\nu^*, \nu)$ is related to the differential scattering cross section of particles of type α with energy E^* by a particle of type i at rest in the following way:

$$d\sigma_i = h_i(\nu^{*}, \nu)\,\delta(\cos\theta_0 - \mu_i)\, d\nu_1\, d\nu_2\, d\nu_3. \tag{6}$$

For instance, isotropic scattering in the center-of-mass system corresponds to

$$h_i(\nu^{*}, \nu) = \frac{\sigma_{st}(\nu^{*})}{4\pi}\,\frac{(m + m_i)^2 - (\nu^{*2}/4c^2)(m - m_i)^2}{m m_i \nu\nu^{*2}}\,\frac{1 - \nu^{*2}/4c^2}{[1 - \nu^2/4c^2]^2}, \tag{7}$$

where σ_{si} is the scattering cross section for particles of type α by those of type i. The integral in Eq. (1) is taken over the region $\nu^* < 2c$.

Equation (1) has been considered in detail elsewhere[2] in treating the scattering by matter of a stream of nonrelativistic particles with $\eta = 0$ and the scattering being isotropic in the center-of-mass system.

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Translated by E. J. Saletan

PROTON COMPONENT OF COSMIC RADIATION AT SEA LEVEL

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MANY studies[1-13] have been carried out on the proton component of cosmic rays at sea level for momenta greater than 0.5 — 0.6 Bev/c. The proton component of the soft radiation has been investigated at sea level only by Filthuth,[14] who considered momenta in the interval 0.2 — 0.9 Bev/c. For momenta greater than 0.6 Bev/c, his results differ from those of many other authors by factors of 1.5 — 2. As has been pointed out by Ogilvie,[10] this indicates a systematic error in Filthuth's measurements. One can suppose, furthermore, that his results are not reliable in the low momentum range which has not been investigated by others.

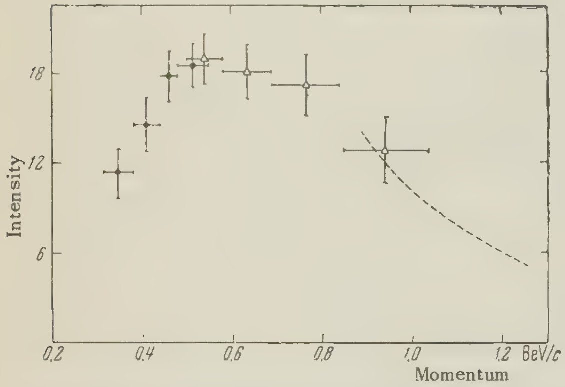
Up to the present time, the lack of adequate and reliable data on the soft proton component of cosmic rays at sea level has been the main obstacle to finding the position of the maximum in the spectrum, and to obtaining the total intensity of the vertical proton current at sea level. The purpose of the work now being reported is to fill in this gap.

The experimental method has been described in detail previously.[12] The measurements lasted 2458 hours. The absolute value of the vertical proton current was measured relative to those particles of the hard component which had ranges greater than 15 cm. Pb. The current of the latter was taken to be $0.785 \times 10^{-2}\,\text{cm}^{-2}\,\text{sec}^{-1}\,\text{sterad}^{-1}$, in accordance with the value adopted earlier.[12]

Momentum interval Bev/c	0.31—0.38	0.38—0.44	0.44—0.48	0.48—0.55
Intensity [$\text{cm}^{-2}\,\text{sec}^{-1}\,\text{sterad}^{-1}\,(\text{Mev/c})^{-1} \times 10^{-6}$]	11.5 ± 1.6	14.5 ± 1.8	17.8 ± 1.6	18.4 ± 1.5

The results of our measurements in various proton momentum ranges from 0.31 to 0.55 Bev/c are shown in the table to the left. These results are plotted in the figure, together with the results obtained previously[12] for momenta from 0.5 to 1.04 Bev/c. It is clear that neighboring points from the present and from the previous sets of data fit together well. The dotted curve is $(1.0 \times 10^{-7})\,p^{-2.8}$, and represents the results of other authors[4,13] in the momentum region p > 1 Bev/c, which we have not investigated. The figure shows that near 0.9 Bev/c, our data agree well with this curve.

For momenta between 0.3 and 0.6 Bev/c, Filthuth's data[14] are 1.5 — 2 times smaller than ours. As mentioned above, a similar discrepancy between Filthuth's data and the data of other authors exists for larger momenta. Our measurements thus support the supposition that there is a systematic error in Filthuth's measurements.



Proton spectrum at sea level. Δ — from Ref. 12; - - - from Refs. 4 and 13; ● — data from the present measurements.

Integrating our spectrum between 0.37 and 1.04 Bev/c, we find that the total intensity of the vertical proton current at sea level in this momentum interval is $I_1 = (1.12 \pm 0.03) \times 10^{-4}\,\text{cm}^{-2}\,\text{sec}^{-1}\,\text{sterad}^{-1}$. We had used the same method and apparatus earlier[15] to investigate the proton spectrum on Alagez mountain 3250 m. above sea level. The intensity of the proton component in the same momentum interval was then $I_2 = (12.15 \pm 0.30) \times 10^{-4}\,\text{cm}^{-2}\,\text{sec}^{-1}\,\text{sterad}^{-1}$. From the ratio of the two, $I_2/I_1 = (10.84 \pm 0.40)$, we can calculate the mean absorption length, which had previously been obtained by various workers using different methods. We obtain $L = (143.5 \pm 6.1)\ \text{g/cm}^2$, which agrees well with the value $L = (140 \pm 10)\ \text{g/cm}^2$ obtained by Mylroi and Wilson,[4] who measured the intensity of the proton component at three heights.

In order to find the total vertical proton current at sea level, we extrapolate the curve shown in the figure to the origin, and use the results of other authors[4,13] for momenta greater than 1 Bev/c. We then find that the total intensity is $I = (1.88 \pm 0.05) \times 10^{-4}\,\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}\,\mathrm{sterad}^{-1}$, which is 2.4% of the number of particles which will penetrate 15 cm Pb.

The maximum of the proton spectrum at sea level is at 0.5 — 0.6 Bev/c. The maximum in the proton spectrum at 3250 m. above sea level is at about the same place.[15] The shapes of the two spectra are the same within experimental error, which is to be expected since for the momenta considered both are the spectra of secondary particles. The shape of the spectrum and the position of the maximum agree well with the theoretical spectrum for secondary protons as computed by Rossi.[16]

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CYCLOTRON RESONANCE IN PLASMA

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THE physical mechanism of cyclotron resonance in metals, for which a theory was formulated in Refs. 1 and 2 leads to the belief that a similar effect should occur in a compensated electron-ion plasma. Let us recall that cyclotron resonance in metals occurs only in a direct magnetic field H parallel to the metal surface* when an electron is repeatedly returned under the influence of the magnetic field at the necessary phase to a skin layer whose thickness δ is small compared with the radius of the electron orbit r in the magnetic field. Naturally, a sufficiently large number of revolutions must be performed in the length of the free path ℓ before resonance can be observed, i.e., $\ell \gg 2\pi r$. Otherwise, resonance is generally absent. Cyclotron resonance occurs when the external field frequency ω is a multiple of the Larmor frequency $\Omega = eH/mc$.

Let us consider the conditions under which resonance in plasma is possible (we neglect ion motion). It is necessary for resonance that "anomalous skin effect" occur:[1,2]

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*Cyclotron resonance was recently observed in tin and bismuth.[3]

$$\delta \ll r \ll l / 2\pi. \tag{1}$$

The inequality to the right is satisfied very well in plasma because of the large free path which, as is known,[4,5] is determined only by the "distant" interactions ($\ell \sim (kT)^2/Ne^4L$, L is the Coulomb logarithm, N the density, e the charge on the electron, T the temperature of the electron gas, and k the Boltzmann constant), by collisions with ions and neutral molecules, or by the characteristic dimensions of the system in the case of bounded plasma.

A sharply pronounced skin effect [the left inequality in (1)] is possible in a plasma because of the negative dielectric constant of the plasma, when the displacement current can be neglected in comparison with the conduction current. Estimates show (see Ref. 2 for example) that the inequality $\delta \ll r$ is equivalent to

$$\bar{v} / c \gg \Omega (m / Ne^2 \omega\tau)^{1/2} \sim (\omega / \omega_0) \cdot (2\pi / \omega\tau)^{1/2}. \tag{2}$$

Here $\bar{v} \sim (kT/m)^{1/2}$ is the average thermal velocity, m the mass, τ the electron free path time, and $\omega_0 = (4\pi Ne^2/m)^{1/2}$ the plasma frequency. We assume that $\omega \approx \Omega$. Thus, for example, inequalities (1), (2) will be satisfied for $T \sim 10^7\,°K$, $N \sim 10^{14}\,cm^{-3}$ and $\omega \sim 10^{10}/sec$ ($\lambda \sim 20\,cm$) if $\ell \gg 1$ cm. If conditions (1) and (2) are satisfied, the entire theory of cyclotron resonance developed in Refs. 1 and 2 for metals can be extended to include plasma. The only difference is that a Maxwellian rather than Fermi equilibrium holds in the plasma: $f_0 = A \exp(-mv^2/kT)$.

Let us cite the final result of a computation of the high-frequency surface impedance Z near resonance for a half-space occupied by plasma:

$$\begin{gathered} Z = R + iX = Z_0 [1 - \exp(-2\pi i\omega / \Omega - 2\pi / \Omega\tau)]^{1/3} \approx Z_0 (2\pi)^{1/3} [1 / \Omega\tau + \pi (\omega - n\Omega)^2 / \Omega^2 + i (\omega - n\Omega) / \Omega]^{1/3}; \\ Z_0 = (\sqrt{3}\,\pi\omega^2 / c^4 B)^{1/3} (1 + i\sqrt{3}); \quad 1 / B = (m / Ne^2)(9\pi kT / 8m)^{1/2}. \end{gathered} \tag{3}$$

The surface impedance is $Z = (4\pi/c)E(0)/H(0)$, where $E(0)$ and $H(0)$ are the alternating electric and magnetic field intensities on the plasma boundaries. Formula (3) is derived under the assumption that the constant magnetic field is strictly parallel to the interface surface,* that the electrons are scattered from the plasma boundaries in an arbitrary manner (but not specularly!), and that the electromagnetic wave is incident perpendicularly on the plasma surface.

At resonance ($\omega \approx n\Omega$, $n \ll \omega\tau/2\pi$), R and X have a minimum, since the external field whose energy the electrons absorb is of fixed intensity. The relative depth of resonance for R/R_0, X/X_0, X/R and the respective resonance fractions are

$$\begin{gathered} \dot{R}_{\text{res}} / R_0 = (2\pi n / \omega\tau)^{2/3}; \quad \Omega_{\text{res}} = (\omega / n)(1 - 1 / \sqrt{2\pi n\omega\tau}); \quad X_{\text{res}} / X_0 = 2 \cdot 3^{-1/2} (\pi n / \omega\tau)^{1/3}; \\ \Omega_{\text{res}} = (\omega / n)(1 + 1 / \omega\tau); \quad (X / R)_{\text{res}} = {}^3/_2 (\omega\tau / \pi n)^{1/2}; \quad \Omega_{\text{res}} = (\omega / n)(1 - 1 / \sqrt{\pi n\omega\tau}). \end{gathered} \tag{4}$$

The small shift in the resonant frequency relative to ω/n is due to the different effective depth of variation in the amplitude and phase of the electric field (see Refs. 1 and 2 for details). The order of magnitude of the relative width of the resonance curves is

$$\Delta H / H \sim |\omega - n\Omega_{\text{res}}| / \omega. \tag{5}$$

Apparently, cyclotron resonance in plasma can be detected by selective absorption of an electromagnetic wave in the resonant region. A "plasma plate" is transparent in a weak magnetic field with $\omega_0 < \omega$ and in a strong magnetic field when (2) is violated, but absorbs strongly in the resonance region.

Moreover, cyclotron resonance permits relaxation time in a plasma to be measured more exactly than usual.

In conclusion, I thank I. M. Lifshitz and Ia. B. Fainberg for discussing the results.

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ON THE RANGE OF MANY-BODY FORCES BETWEEN A Λ^0-HYPERON AND NUCLEONS

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THE forces between a Λ^0-hyperon and one nucleon [$(\Lambda^0 - N)$ forces], admissible on grounds of isotopic invariance, have a range several times shorter than the usual nuclear forces.[1,2] Thus, the forces suggested by the experiments on pair production and due to virtual processes with K-meson exchange

$$\Lambda^0 + N \rightarrow N' + \overline{K} + N \rightarrow N' + \Lambda^{0\prime}, \tag{1}$$

have a range $1/m_K$, i.e., a range about three times shorter than the radius of nucleon-nucleon forces, which is of order $1/m_\pi$. However, isotopic invariance allows an interaction in which the Λ^0-hyperon exchanges with the nucleon two (in general, an even number of) π-mesons:

$$\Lambda^0 + N \rightarrow \Lambda^{0\prime} + \pi + \pi + N \rightarrow \Lambda^{0\prime} + N'. \tag{2}$$

One can also conceive of similar 2π-meson forces involving Σ-hyperons:[3]

$$\Lambda^0 + N \rightarrow \Sigma + \pi + N \rightarrow \Sigma + N' \rightarrow \Lambda^{0\prime} + \pi + N' \rightarrow \Lambda^{0\prime} + N'. \tag{3}$$

The forces due to (2) and (3) have a range $1/2m_\pi$.

The existence of Λ^0-hyperfragments indicates that the $(\Lambda^0 - N)$ forces are sufficiently strong to lead to binding of a Λ^0-hyperon to nucleons. In this connection it is interesting to determine whether there exist $(\Lambda^0 - N)$ forces with the usual range $1/m_\pi$.

Such forces indeed are possible if we consider the interaction of the Λ^0-hyperon with two nucleons instead of one. Thus, the 2π-meson forces between a Λ^0 particle and two nucleons, realizably by virtual processes of the form

$$\Lambda^0 + N_1 + N_2 \rightarrow \Lambda^{0\prime} + \underbrace{\pi + \underbrace{\pi + N_1}_{} + N_2}_{} \rightarrow \Lambda^{0\prime} + N_1' + N_2', \tag{4}$$

have a range $1/m_\pi$, since the Λ^0-hyperon exchanges with each of the nucleons only one π-meson. This conclusion regarding the range of the forces is purely qualitative and follows from the uncertainty relation for the energy and time, and actually involves no approximations.

As an illustration it is easy to calculate the potential of the $(\Lambda^0 - N)$ forces of the type (4) in the static approximation. This potential contains the product of the exponentials $\exp(-m_\pi r_{i\Lambda}) \exp(-m_\pi r_{2\Lambda})$, where r_i is the distance between the Λ-hyperon and the i-th nucleon. Thus many-body forces of range $1/m_\pi$ between a Λ^0-hyperon and nucleons are possible, in contrast to two-body $(\Lambda^0 - N)$ forces, whose range is less. One can therefore assume that the suggested many-body forces may play an important role in the interaction of the Λ^0-hyperon with nucleons.

This hypothesis could be checked by measuring the cross section for the scattering of a slow Λ^0 particle by a proton and by some simple nucleus such as deuterium or helium. If the present hypothesis is true the cross sections should be considerably larger for scattering by nuclei than for scattering by protons.

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ROTATIONAL BANDS OF EVEN-EVEN AXIALLY SYMMETRIC NUCLEI

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It has been shown by Davydov and Filippov[1] that by using the Hamiltonian obtained by Ford[2] in averaging the interaction between the external nucleons and the nuclear core, one can write the equation for the collective motion of an axially symmetric even-even nucleus with total angular momentum ħJ in the form

$$d^2U_\nu / d\zeta^2 - 2\zeta dU_\nu / d\zeta + 2\nu U_\nu = 0, \qquad (1)$$

where U_ν satisfies the boundary condition

$$U_\nu(-\delta\xi) = 0, \quad U_\nu(\zeta)\, e^{-\zeta^2/2} \to 0, \qquad \text{for } \zeta \to \infty. \qquad (2)$$

The eigenvalue ν of Eq. (1) is not in general an integer, and determines the energy $\epsilon_\nu(J)$ of the collective nuclear motion by the equation

$$\varepsilon_\nu(J)/\hbar\omega_0 = (\nu + {}^1/_2)\sqrt{1 + J(J+1)/\delta^4\xi^4} + J(J+1)/6\delta^2\xi^2 + {}^1/_2\,\delta^2(\xi - 1)^2, \qquad (3)$$
$$\xi^3(\xi - 1) = J(J+1)/3\delta^4.$$

Thus the energy of nuclear collective motion for each value of J = 0, 2, 4, . . . is determined uniquely by just the two parameters ω_0 and δ, which are related to parameters of Bohr and Mottelson's[3] generalized nuclear model by the expressions $\omega_0 = \sqrt{C/B}$, $\delta = \beta(BC/\hbar^2)^{1/4}$.

Davydov and Filippov[1] investigated the solution of Eq. (1) for the case $\delta \leq 1$. In this note we present the results of a solution of this set of equations for the case $\delta > 1$.

The figure gives a graph of $\epsilon_\nu(J)/\hbar\omega_0$ vs. δ; the numbers on the curves give the values of J. It is seen from the figure that when $\delta > 2.5$, the energy spectrum of collective excitations of even-even nuclei breaks up into a set of rotational-vibrational bands. In Table 1 we present a comparison of the theoretical excitation energies of the first and second rotational band for certain nuclei with the experimental data. We also give the values of $\hbar\omega_0$ and δ which have been used to calculate the theoretical excitation energy.

TABLE 1

Nucleus and literature reference	J	Energy level (kev)		$\hbar\omega_0$ (kev)	δ
		Theory	Experiment		
W^{182} [4,5]	2	100.09	100.09	1101	3.48
	4	320.3	329.36		
	6	641.6	677.6		
	0	1101	—		
	2	1222	1222		
	4	1481	1488.6		
Th^{232} [6]	2	50	50	710	3,93
	4	163	165		
	6	332	—		
	0	710	—		
	2	770	770		
	4	901	—		
U^{234} [5]	2	43	43	803	4.48
	4	141	142		
	6	290	295		
	0	803	803		
	2	855	—		
	4	966	—		
Pu^{238} [5]	2	44.2	44.2	935	4.73
	4	147.7	146		
	6	304.8	303		
	0	935	935		
	2	986	986		
	4	1100	1073		

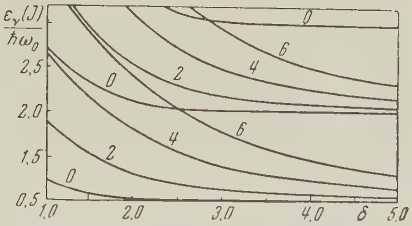
In Table 2 we give the δ dependence of the energy ratios of the first and second (1 and 2) rotational-state sublevels in the first and second (I and II) bands of the rotational states of the nucleus.

If the energy of collective oscillations is approximated in the form

$$E_J = n\hbar\omega_0 + AJ(J+1) - BJ^2(J+1)^2, \qquad A = \hbar^2/2I, \qquad B = a(\hbar\omega_0)^{-2}(\hbar/I)^3, \qquad (4)$$

TABLE II

δ	1,0	1,5	2,0	2,5	3.0	3,5	4,0	4.5	5.0
$\varepsilon_{1II}/\varepsilon_{1I}$	1.48	1,39	1.36	1,31	1.26	1,21	1,16	1,13	1,11
$\varepsilon_{2I}/\varepsilon_{1I}$	2.17	2,38	2.70	2,87	3.02	3,21	3,27	3,29	3,33
$\varepsilon_{2II}/\varepsilon_{1II}$	1.76	2.16	2,43	2,52	2.94	3.16	3,25	2,27	3,31



it follows from Table II that the moment of inertia I of the nucleus in the second rotational band is less than in the first. This decrease of I is greater for lower values of δ. The quantity a, which determines the coupling of rotational and vibrational states in Eq. (4), is greater in the second rotational band than in the first. Thus if one were to use Eq. (4) to describe collective oscillations, one would need five parameters, rather than the two that are needed to solve (1) with Eq. (3).

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MESONIUM AND ANTIMESONIUM

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GELL-MANN and Pais[1] were the first to point out the interesting consequences which follow from the fact that K^0 and $\widetilde{K}^0$ are not identical particles.[2] The possible $K^0 \rightarrow \widetilde{K}^0$ transition, which is due to the weak interactions, leads to the necessity of considering neutral K-mesons as a superposition of particles K_1^0 and K_2^0 having a different combined parity.[3] In the present note the question is treated whether there exist other "mixed" neutral particles (not necessarily "elementary") besides the K^0-meson, which differ from their anti-particles and for which the particle → antiparticle transitions are not strictly forbidden.

The laws of conservation of the number baryons and light fermions (or as sometimes called, conservation of nucleon[4] and neutrino[5] charge) strongly limit the number of possible mixed neutral particles. Because of the first-mentioned law mixed particles cannot occur amongst the baryons (e.g. a neutron; a hydrogen atom etc.), and because of the second law such particles cannot exist among the light particles with only one fermion (e.g. neutrino, the systems π^+e^- and π^-e^+, etc.).

From this it evidently follows that besides the K^0-meson the only system consisting of presently-known constituents which could be a mixed particle would be mesonium, defined as the bound system (μ^+e^-). Antimesonium, i.e., the system (μ^-e^+), clearly is different from mesonium and, furthermore, the mesonium → antimesonium inversion is not only not forbidden by any of the known laws, but actually should occur by virtue of already established interactions.

Indeed, the transitions

$$(\mu^+e^-)\rightarrow(\nu+\tilde{\nu})\rightarrow(\mu^-e^+) \tag{1}$$

would be induced by the same interaction that is responsible for the decay of the μ-mesons. The probability $1/\theta$ of the real decay process

$$(\mu^+e^-)\rightarrow\nu+\tilde{\nu}+106.1\ \text{Mev}, \tag{2}$$

which can be easily obtained by taking into account the size of the mesonium, is found to be $10^{-4}\ \text{sec}^{-1}$, i.e. this probability is 10^{10} times smaller than the usual decay probability of the μ-meson. It is therefore impossible in practice to observe this process which would be indicated by a track corresponding to a stopping μ^+-meson which decays without the emission of a decay electron.

The inversion time, process (1), is proportional to $\hbar/c^2\Delta m$ and is determined[1,6] by the mass difference Δm between the systems symmetrical and antisymmetrical with respect to mesonium and antimesonium. This mass difference is proportional to the first power of the matrix element of the mesonium → antimesonium transition. If this transition is due to a process involving two consecutive transitions, as in (1), Δm is proportional to the square of the coupling constant. The inversion time is then of the order θ and 10^{10} times longer than the half life of the π-meson ($\tau = 2\times10^{-6}$ sec.) The meson half life then determines also the mesonium half life.

However, if one admits a direct $(\mu^+e^-)(\mu^-e^+)$ interaction, then the inversion time T can be very significantly shorter than θ. Indeed, then the mass difference $\Delta m'$ between the symmetric and the antisymmetric system ($\Delta m' \approx 2M/c^2$ where M is the transition matrix element) is proportional to the first power of the coupling constant g. We consequently have:*

$$T\sim\hbar/c^2\Delta m'\sim\hbar/(2g/\pi r^3),$$

where r is the radius of the mesonium. Assuming that the direct interaction $(\mu^+e^-)(\mu^-e^+)$ has the same strength as the other weak interactions, we get $g\approx3\times10^{-49}\ \text{erg}\,\text{cm}^3$ and $T\approx5\times10^{-4}$ sec, which is only 300 times longer than τ. Under these conditions it seems at first glance that the mesonium → antimesonium inversion should be observable without too great difficulty. For example, one should see a "fast" negative electron after stopping a μ^+ meson according to the process $(\mu^+e^-)\rightarrow(\mu^-e^+)\rightarrow e_{\text{fast}}+\nu+\tilde{\nu}+e^+$. Unfortunately, however, the inversion mesonium → antimesonium cannot take place inside of matter, owing to the electrical charge asymmetry of the nucleons. This leads to a difference of the mass of mesonium and antimesonium in matter. Furthermore, it should be pointed out that the probability of emission of a fast negative electron (in vacuo) is proportional to $(\tau/T)^2$ and not to (τ/T). Denoting by $\epsilon_{\mu^+}(t)$ and $\epsilon_{\mu^-}(t)$ the probability of finding a mesonium and antimesonium respectively in vacuo at time t if one mesonium "atom" exists at time zero, then

$$\varepsilon_{\mu^+}(t)\sim{}^1/_2\,e^{-t/\tau}\left(1+\cos\frac{t}{T}\right),\quad \varepsilon_{\mu^-}(t)\sim{}^1/_2\,e^{-t/\tau}\left(1-\cos\frac{t}{T}\right),$$

where the half life of mesonium and antimesonium was assumed to be the same and equal to the μ-meson half life. For these initial conditions the emission probability of a positive and negative electron respectively in vacuo is given by

$$P(e^+)\sim\int_0^\infty\varepsilon_{\mu^+}(t)\frac{dt}{\tau}\sim\frac{1}{2}\left(1+\frac{T^2}{T^2+\tau^2}\right)\sim1,\qquad P(e^-)\sim\int_0^\infty\varepsilon_{\mu^-}(t)\frac{dt}{\tau}\sim\frac{1}{2}\left(1-\frac{T^2}{T^2+\tau^2}\right)\sim\frac{1}{2}\left(\frac{\tau}{T}\right)^2.$$

If there would exist in nature charged particles of long half life with small nuclear interaction, then an effect analogous to the one presently described could be observed. The half life of the particles of mass $\sim500\ m_e$ observed by Alikhanian et al.[8] has not been determined yet; it is merely known to be much greater than 5×10^{-9} sec.

It was assumed above that there exists a conservation law for the neutrino charge, according to which a neutrino cannot change into an antineutrino in any approximation. This law has not yet been established; evidently it has been merely shown that the neutrino and the antineutrino are not identical particles.[9] If the two-component neutrino theory[10] should turn out to be incorrect (which at present seems to be rather improbable) and if the conservation law of neutrino charge would not apply, then in principle neutrino → antineutrino transitions could take place in vacuo. Even in this case, as well as in the case where one assumes that to every world there exists an antiworld, the number of neutrinos and antineutrinos in the universe would have to be the same.

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*The analogous case of the $K^0\rightarrow\tilde{K}^0$ transition to first order in the weak interaction has been treated in detail in Ref. 7.

The author is grateful to Ia. B. Zel'dovich, L. B. Okun', and I. G. Ivanter for critical remarks and interesting discussions.

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ON THE POSSIBILITY OF $\pi \rightarrow e + \nu + \gamma$ DECAY

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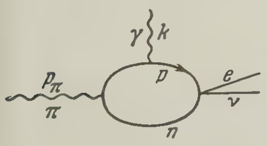
In the presence of the strong interaction of π mesons with nucleons, the decay $\pi \rightarrow e + \nu$ might occur on account of the β-decay interaction, with a lifetime of the same order as that of the decay $\pi \rightarrow \mu + \nu$. In reality, however, the β-decay interaction contains scalar (S) and tensor (T) terms. The decay $\pi \rightarrow e + \nu$ obviously cannot occur on account of the T type term. It may appear that with nonconservation of parity[1] such a decay could take place on account of the S type term, but in reality experiment indicates the presence of the terms not conserving parity in the lepton part of the interaction, whereas for the absence of the $\pi \rightarrow e + \nu$ decay it is sufficient that the heavy particle interaction not contain pseudoscalar terms, since under this condition the nucleon loop reduces to zero.



It seems difficult, however, to explain the absence of the decay $\pi \rightarrow e + \nu + \gamma$. Such a decay can occur on account of the tensor interaction if the virtual nucleon emits the γ-quantum.[4] We estimate the probability of such a process in the first order of perturbation theory in terms of G, where G is the constant of the π-meson-nucleon interaction $G(\tau \Phi_\pi)(\overline{\psi}_N \gamma_5 \psi_N)$.

The simplest diagram for the $\pi \rightarrow e + \nu + \gamma$ decay is shown in the figure. With e in rationalized units and h = c = 1, we get for the matrix element

$$M = g_T e (2k \cdot 2m_\pi)^{-1/2} \sqrt{2}\, G I_{\varepsilon\mu} (\psi_e \gamma_5 \gamma_\varepsilon \gamma_\mu \psi_\nu)\, \delta (p_\pi - p_\nu - p_e - k),$$

where $I_{\epsilon\mu}$ is a logarithmically divergent integral over the variables of the virtual nucleons,

$$I_{\varepsilon\mu} = \mathrm{Sp}\int d^4p\,\frac{i\hat{p}-M}{p^2+M^2}\,\hat{e}\,\frac{i(\hat{p}-\hat{k})-M}{(p-k)^2+M^2}\,\gamma_5\gamma_\varepsilon\mu_\mu\,\frac{i(p-p_\pi)-M}{(p-p_\pi)^2+M^2}\,\gamma_5.$$

Here M is the mass of the nucleon, e and k are the polarization and four-dimensional momentum of the γ-quantum, and the other symbols are standard. Taking account of only the logarithmically divergent term we have

$$I_{\varepsilon\mu} \approx 2\pi^2 F_{\varepsilon\mu} G \ln(\Lambda^2/M^2); \quad F_{\varepsilon\mu} = e_\varepsilon k_\mu - e_\mu k_\varepsilon.$$

For the probability of the $\pi \to e + \nu + \gamma$ decay we get

$$dW = g_T^2\left(\frac{e^2}{4\pi}\right)\left(\frac{G\sqrt{2}}{4\pi^2}\ln\frac{\Lambda^2}{M^2}\right)^2\frac{1}{(2\pi)^2}\,4m_\pi\left(\frac{m_\pi}{2}-q\right)\left(q+k-\frac{m_\pi}{2}\right)dp\,dq.$$

Here q and k are the momenta of the electron and the γ-quantum; we neglect the mass of the electron.

For the total probability we find

$$W = \frac{1}{96}\frac{1}{4\pi^2}\left(\frac{e^2}{4\pi}\right)\left(\frac{G\sqrt{2}}{4\pi^2}\ \ln\frac{\Lambda^2}{M^2}\right)^2 g_T^{02}\left(\frac{m_\pi}{m_e}\right)^4 m_\pi.$$

Substituting

$$g_T = 1.8\cdot10^{-49}\ \mathrm{erg/cm^3} = g_T^0/m_e^2,\ g_T^0 = 3.8\cdot10^{-12},\ \ln\Lambda^2/M^2 \sim 1,\quad G^2/4\pi \sim 10$$

and going over to ordinary units, we get $W \sim 0.5 \times 10^4\ \mathrm{sec}^{-1}$. The ratio of the decay probabilities is

$$(\pi\to e+\nu+\gamma)/(\pi\to\mu+\nu) = \rho_\gamma \sim 0.5\cdot10^4/0.4\cdot10^8 \sim 10^{-4} \qquad (\tau(\pi+\mu+\nu) = 2.5\cdot10^{-8}\,\mathrm{sec}).$$

The ratio ρ_γ has been estimated by Treiman and Wyld.[4] In doing so they assumed that the decay $\pi \to \mu + \nu$ goes through a nucleon loop and a pseudovector interaction of the β-decay type. The logarithmically diverging integral for the nucleon loop then has the form $I_\nu = p_\nu M f_p$, where p is the momentum of the meson, M is the mass of the nucleon, and f_p is a dimensionless factor. Similarly the corresponding part of the matrix element of the $\pi \to e + \nu + \gamma$ decay was taken in the form $I_{\mu\nu} = e_\mu k_\nu f_T$, based on considerations of dimensional and gauge invariance. Then $\rho_\gamma \sim (g_T f_T/g_p f_p)^2$, and it was assumed that $f_p \sim f_T$ and $g_T/g_p > 50$. This last estimate is based on the assumption that the values of g_p for the meson and the electron are the same, and then β-decay gives $g_p < g_T/50$. With this approach, $\rho_\gamma \sim 1/500$. In Ref. 4 it is shown that the correct lifetime of the π meson is obtained for $g_p \sim 10^{-49}\,\mathrm{erg/cm^3}$, $f_p \sim 0.1$. But $g_p \approx g_T/50$ gives $g_p \approx (1/25)\,10^{-49}\,\mathrm{erg/cm^3}$ and it is necessary to take $f_p \sim 2.5$ in order to get the same lifetime. In our work the lifetime of the $\pi \to \mu + \nu$ decay is taken from experiment; the value of f_T corresponds to $(G \times 2^{1/2}/4\pi^2)\ln(\Lambda^2/M^2) \sim 0.4$.

At the Seventh Rochester Conference Cassels reported that experiment gives $\rho_\gamma < 2 \times 10^{-6}$.

I express my gratitude to Prof. I. Ia. Pomeranchuk for suggesting the problem and for discussions, and to Iu. A. Simonov for valuable remarks.

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THE POSSIBLE IDENTIFICATION OF $\overline{\Sigma^-}$ ANTIHYPERONS IN EMULSION

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IN studies of the interaction of K-mesons with nuclei[1] it was established that $m(\Sigma^-) - m(\Sigma^+) = (15.9 \pm 2.5)\,m_e$. This mass difference may be used to identify the antihyperon $\overline{\Sigma^-}$, which is expected to decay by $\overline{\Sigma^-} \to \overline{n} + \pi^+$. From the mass difference between $\overline{\Sigma^-}$ and Σ^+ it follows that the π^+-mesons in $\overline{\Sigma^-}$ and Σ^+ decay should have different energy and different range in emulsion (10.1 and 9.1 cm respectively). The π-meson energies were calculated from the data of Shapiro.[2] The ranges are taken from the curves of Baroni et al.[3]

The ionization straggling of ranges in this energy-region is about 1 per cent. A π-meson range of 10 cm can be measured to an accuracy of 1.5 per cent. The expected difference in π-meson ranges is 10 per cent. These numbers show that identification of a $\overline{\Sigma^-}$ antihyperon is possible if the decay occurs at rest and if the π^+-meson stops in the emulsion. In order to make sure that the decay was at rest, one should look for cases in which the π^+-meson comes out backward in relation to the direction of the primary.

I thank M. I. Podgoretskii for criticism of this work.

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STATISTICAL TREATMENT OF ELEMENTARY PARTICLE STRUCTURE

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IN present-day quantum field theory, every charge interacts with a field of elementary particles in its neighborhood. For electric charge the particles are photons or pseudophotons, for nucleonic charge they are mesons, and so on. The problem of elementary particle structure is intimately connected with these fields. The question arises, whether these fields may exhibit some of the properties of ordinary particles, particularly the statistical properties.

We consider an elementary particle as a cloud of virtual particles obeying canonical statistics. The distribution of pseudophotons (with energy $\epsilon = \hbar\omega$) is given by

$$u(\omega)\,d\omega = (\hbar\omega^3(2\pi^2c^3)\,[\exp(\hbar\omega/\Theta + \alpha) - 1]^{-1}\,d\omega; \tag{1}$$

For black-body radiation $\alpha = 0$, but in general $\alpha \neq 0$. The uncertainty principle connects the energy of a virtual particle with its duration t and its distance R from the center as follows,

$$\varepsilon t \sim \hbar;\quad R \sim ct;\quad \varepsilon = \hbar c/pR; \tag{2}$$

Here c is the light-velocity and p is a factor of order unity. In other words, the uncertainty principle protects us from any contradiction with energy-conservation, if we suppose that a virtual particle moving with velocity $\sim c$ and having energy $\epsilon = \hbar\omega$ can be observed at distances ranging from zero to $R \sim \hbar c/\epsilon$.

At a point at a distance r one may find particles with energies from zero to $\hbar c/r$. The total energy-density at distance r is thus

$$W(r) = \int_0^{r/c} u(\omega)\,d\omega = -\frac{\hbar c}{2p^4\pi^2}\int_\infty^r \left[\exp\left(\frac{\hbar c}{p\Theta R} + \alpha\right) - 1\right]^{-1} \frac{dR}{R^5}. \tag{3}$$

where we have changed the variable of integration from ω to c/R.

The total energy E of the virtual field is the integral of W over all space. This energy we identify with the rest-energy $m_e c^2$ of the particle, supposing that other virtual fields such as the gravitational field are negligible. The energy-density $W(r)$ is then the energy-density of the Coulomb field. The integrals W and E are everywhere convergent. For large r, Eq. (3) gives

$$W(r) = 1/8p^4\pi^2(e^\alpha - 1)\,r^4,$$

which must be compared with the usual expression $W(r) = e^2/8\pi r^4$ for the energy-density of the electrostatic field of a charge e. Hence we have $\hbar c/e^2 = \pi p^4(e^\alpha - 1)$. If $p \sim 1$, $\alpha \sim 1$. On the other hand, the total energy is

$$E = m_e c^2 = \frac{2\Theta}{p^3\pi}\int_0^\infty \frac{dz}{z^4}\int_0^z \frac{z^3\,dz}{e^{z+\alpha} - 1},$$

which gives for $p \sim 1$

$$\Theta_e \sim e^\alpha m_e c^2 \sim 10^{-4}. \tag{4}$$

This value of Θ is comparable to that which appears in the thermodynamical theory of multiple particle production,[1] $\Theta \sim m_\pi c^2 \sim 10^{-4}$. Significant deviations from the Coulomb potential should begin when $\hbar c/p\Theta R_0 \sim 1$, $R_0 \sim \hbar c/\Theta \sim 10^{-13}$ cm. The maximum energy density occurs at $r = 0$ and is given by Eq. (3)

$$W_{\max} \sim \Theta^4/e^\alpha(\hbar c)^3 \sim m_e^4 c^8/e^6 \sim 10^{32}. \tag{5}$$

For the nucleon, the virtual particles are π-mesons with mass m_π. Their energy cannot be less than $m_\pi c^2$, and their range of action cannot exceed $R_\pi = \hbar/m_\pi c = 1/k_\pi$. This argument, as is well known,[2] leads to an estimate of the range of nuclear forces. By analogy with Eq. (3), we have

$$W(r) = -\frac{\hbar c}{2p^4\pi^2}\int_{R_\pi}^r \left[\exp\left(\frac{\hbar c}{p\Theta R} + \alpha\right) - 1\right]^{-1} \frac{dR}{R^5};\quad E = 4\pi\int_0^{R_\pi} W(r)\,r^2\,dr. \tag{6}$$

Here α and Θ will in general have new values; m_n is the nucleon mass. The integrals (6) also converge. If we compare Eq. (6) with the Yukawa potential $\varphi = g\,e^{-k_\pi r}/r$, we find

$$\Theta_n \approx [\pi g^2 k_\pi^4 \hbar^3 c/p^3 m_n]^{1/3} \sim 10^{-4}, \tag{7}$$

which agrees with Eq. (4). This gives us some reason to make the hypothesis, $\Theta_n \sim \Theta_e$. The maximum energy-density at the center is

$$W_{\max} \approx (\Theta^4/\pi^2 e^\alpha)(\hbar c)^{-3}\exp(-m_\pi c^2/\Theta) \sim 10^{32},$$

which also agrees with Eq. (5).

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PROCESSES INVOLVING ANTIHYPERONS

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THE 10 Bev proton accelerator, which will soon start operating in the Soviet Union, will make it possible to study experimental phenomena involving antihyperons. We here discuss some processes which may profitably be investigated when antihyperons become available.

1. GENERAL REMARKS

To explain the decay processes of some heavy mesons and the transmutation of one heavy meson into another (Lee-Orear theory) without postulating a direct coupling of these mesons with each other or with the pion field, one is forced to introduce the virtual creation and annihilation of baryon-antibaryon pairs. In fast processes there is conservation of strangeness, and therefore a heavy meson with $S \neq 0$ (for example a K-particle) cannot produce a nucleon-antinucleon pair. Nucleon-antihyperon or hyperon-antinucleon pairs can simultaneously conserve strangeness and the difference between baryon and antibaryon number. The latter conservation law makes it impossible to identify antihyperons with hyperons even when they are neutral.

The analysis of antihyperon creation and interaction processes can yield useful information about the quantum numbers (isotopic spin, z-component of isotopic spin, strangeness, etc.) usually assigned to antiparticles. The chief aim of the following discussion is to determine the consequences of the conservation of a chosen system of quantum numbers in the most important fast processes involving antihyperons.

To assign a definite set of quantum numbers to antihyperons and to analyze their creation and interaction processes, we must start from the following assumptions: (a) The total isotopic spin and its z-component are conserved in all fast processes involving antihyperons $\overline{H}$, nucleons N, heavy mesons K and pions; the strangeness is then also conserved. (b) The z-component of isotopic spin has equal magnitude and opposite sign for each particle and its antiparticle. (c) The strangeness has equal magnitude and opposite sign for particle and antiparticle.

Assumption (a) is connected with the notion that all the interactions responsible for the processes here considered are charge-independent. Assumptions (b) and (c) follow from the formal relations[1] between T_3, the charge conjugation operator C, and the strangeness S.

2. CREATION PROCESSES

The physically allowed creation processes are limited by the conservation of the strangeness which is assigned to antibaryons by assumption (b). The study of direct production of hyperon-antihyperon pairs in nucleon-nucleon collisions is highly inconvenient; the short life-time of hyperons (and presumably of antihyperons) makes it difficult to extract a beam of them from the accelerator; and in any case we should obtain a mixture of hyperons and antihyperons. Moreover, the thresholds of direct processes are rather high (around 7 Bev for production of $\Lambda-\overline{\Lambda}$ pairs on nucleons at rest).

It would be much better to use the indirect process

$$K + N \rightarrow \overline{H} + N + N. \tag{1}$$

This process can occur in a K-meson beam outside the accelerator, and it produces only antihyperons. The threshold is about 4 Bev for the K-meson kinetic energy, and K-mesons of this energy can be produced by protons with kinetic energy a little over 5 Bev. If process (1) occurs in a K-meson beam passing through a liquid hydrogen bubble-chamber, the interaction and decay of antihyperons can also be observed in the chamber. A rough estimate indicates that a significant number of $\overline{H}$ can be obtained from a primary 10 Bev proton beam of intensity equal to that of the proton beam in the bevatron.

3. INTERACTION PROCESSES

We predict that the strong interaction between antihyperon and nucleon will give rise to the following "fast" process,

$$\overline{H} + N \to K + \pi. \qquad (2)$$

In reaction (2), $\overline{\text{K}}$ cannot be produced in the absorption of $\overline{\text{H}}$. In general, $\overline{\text{K}}$ cannot be created except in processes where a $\text{K}-\overline{\text{K}}$ pair appears together with the K. If $\overline{\text{H}}$ is captured into excited fragments, as H is, then a characteristic feature of the annihilation of these fragments will be the absence of $\overline{\text{K}}$.

In addition to (2), there will be charge-exchange processes and processes of the type

$$\overline{\Sigma} + N \to \Lambda^0 + N; \qquad (3)$$

These must be considered probable, in view of the observed production of antineutrons by charge-exchange between antiprotons and nucleons.[2]

Next we consider in more detail the interactions of $\overline{\Sigma^+}$. Being negative, this particle after losing energ by ionization and stopping, will be captured in a nucleus or in hydrogen. As an example we consider the interaction with a deuteron. The deuteron having zero isotopic spin, the initial state is an eigenstate of isotopic spin. Among the possible reactions are

$$\overline{\Sigma}^+ + d \to n + \pi^- + K^+, \qquad (4a)$$

$$n + \pi^0 + K^0, \qquad (4b)$$

$$p + \pi^- + K^0. \qquad (4c)$$

In the initial state T = 1. The $\pi - \text{N}$ resonance will strongly influence those final configurations in the reactions (4) which have $\text{T} = \frac{3}{2}$. For such configurations, the ratios between the cross-sections for the three processes will be given by

$$\sigma(a) : \sigma(b) : \sigma(c) = 9 : 2 : 1 \qquad (5)$$

and the corresponding K^+/K^0 ratio will be 3. Recent experiments indicate a strong interaction in the $\text{K}-\text{N}$ system with T = 1. Because of the high energy of the products of the reactions (4), it is possible that the strong $\text{K}-\text{N}$ interaction with T = 1 will dominate the effect of the $\pi - \text{N}$ resonance. In the limiting case of a pure $\text{K}-\text{N}$ final state with T = 1, the ratios of the cross-sections will be

$$\sigma(a) : \sigma(b) : \sigma(c) = 1 : 2 : 1; \qquad (6)$$

and the K^+/K^0 ratio will be $\frac{1}{3}$.

The $\overline{\text{H}} - \text{N}$ annihilation may give information concerning the existence of two K-particles of opposite parity or the alternative possibility that only one K-particle exists with non-conservation of parity in its decay. In the latter case, the ratios between the decay modes τ and θ will be the same as for heavy mesons produced in other processes ($\pi - \text{N}$ or $\text{N}-\text{N}$ collisions, etc.).

If there are two heavy mesons with opposite parity, the ratios between the decay modes should vary. If an annihilation takes place from a state of definite total and orbital angular momentum, only one of the K-mesons should be observed in the annihilation products.

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